EESC 9945

Geodesy with the Global Positioning System

Class 5: *Effects of Atmospheric Propagation*

Signal Propagation

- Both the pseudorange and phase models have the term $\tau_r^s(t)$, the time it takes the signal to propagate from GPS satellite s (at the point of transmission) to the receiver r

- We had been assuming that $\rho_r^s(t) = c\tau_r^s(t)$, where c is the speed of light in a vacuum and $\rho_r^s(t)$ is the geometric distance (range) from the point of transmission to the point of reception

- In fact, the atmosphere of Earth effects the propagation of the signal and must be accounted for

Electromagnetic Wave Propagation

- Maxwell Equations for free space

$$\nabla \cdot \vec{E} = 0 \quad \nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} + \frac{\mu\epsilon}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

  $\vec{E}$, $\vec{B}$ are electric, magnetic fields; c is speed of light in vacuum; ϵ is dielectric constant; μ is magnetic susceptibility

- Combining curl equations, using zero divergences yield wave equations for $\vec{E}$ and $\vec{B}$ of form $\nabla^2 u - \frac{\mu\epsilon}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

Electromagnetic Wave Propagation

- For plane waves traveling in z direction, solutions are B and $E \sim \exp(kz - \omega t)$ with $\omega\sqrt{\mu\epsilon} = k = 2\pi/\lambda$

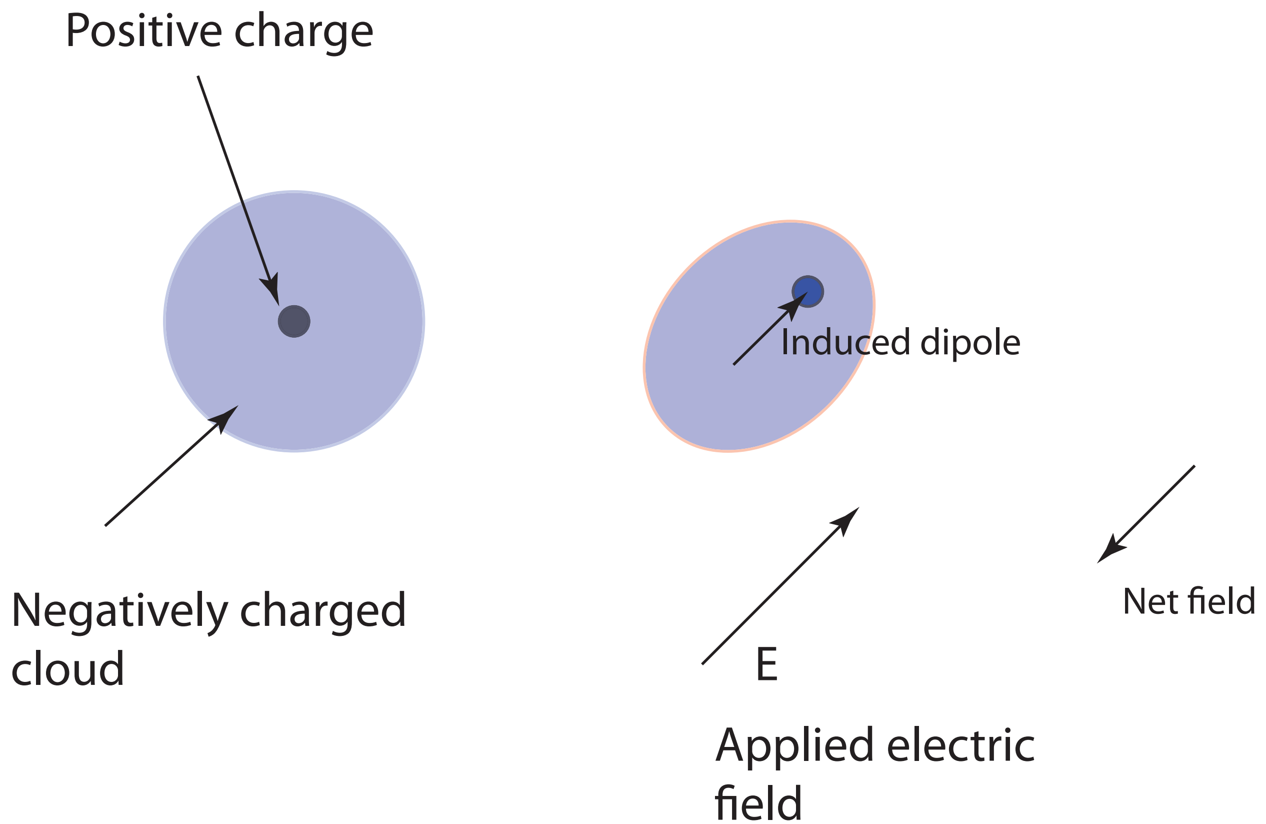
- Constant phase means $kz - \omega t = \text{constant}$

- $\frac{d}{dt}$ yields **phase velocity** $v_p = \frac{dz}{dt} = \frac{\omega}{k} = \lambda f = \frac{c}{\sqrt{\mu\epsilon}}$

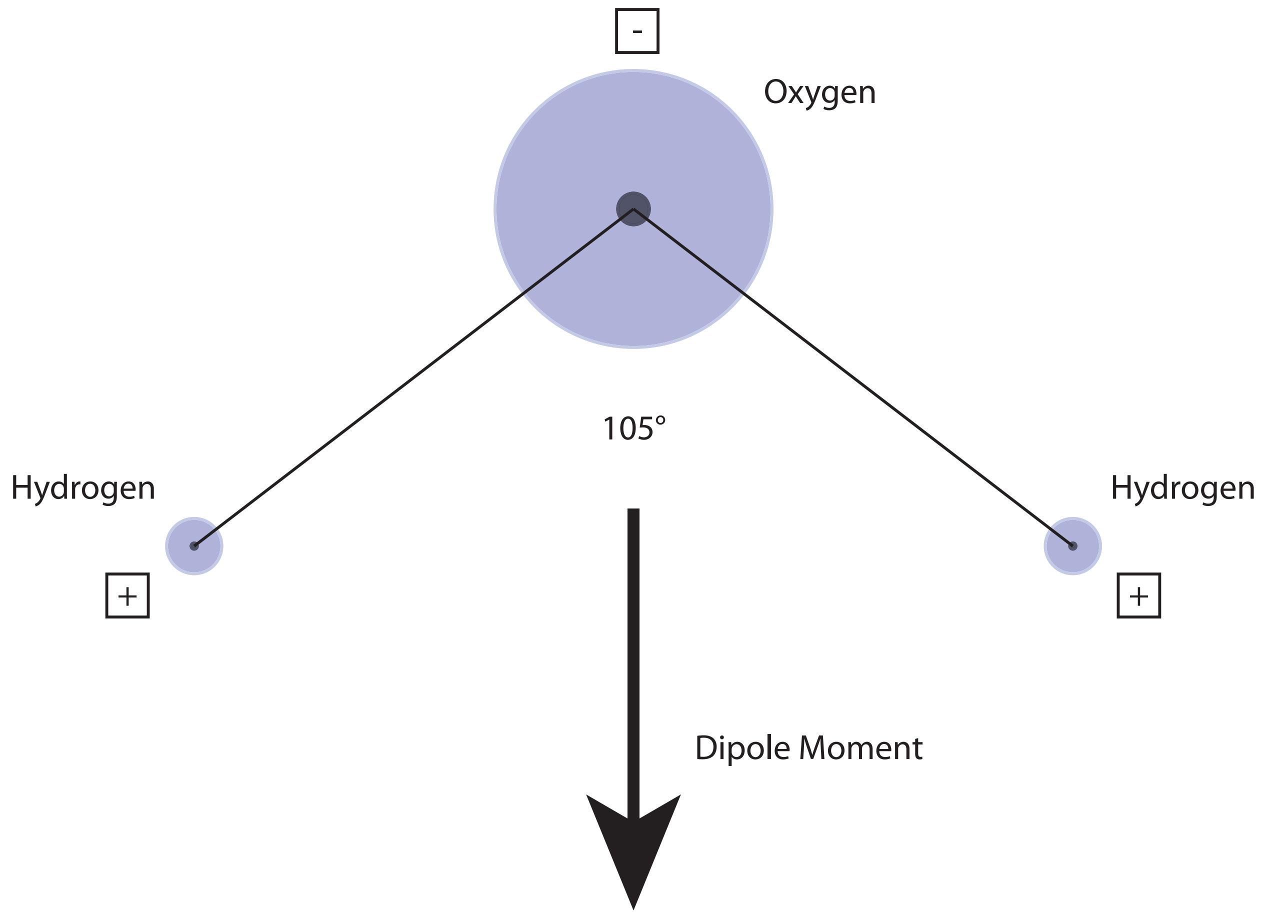
- For the atmosphere, $\mu \simeq 1$

- $\epsilon > 1$ due mainly to induced and permanent electric dipoles

Induced Dipole Moment



Permanent Dipole Moment of Water (Vapor)



Electric Susceptibility χ & Refractive Index n

- χ relates application of weak electric field $\vec{E}$ to polarization per unit volume $\vec{P}$: $\vec{P} = \chi\vec{E}$

- Related to dielectric constant for isotropic medium

  $\epsilon = 1 + 4\pi\chi$

- Phase velocity $v_p = \frac{\omega}{k} = \frac{c}{\sqrt{\mu\epsilon}} = \frac{c}{n}$

- n may be < 1 or > 1 or complex ($= 1$ for vacuum)

- Imaginary part means absorption (we'll ignore)

Group Velocity

- If signal not monochromatic:

$$u(z,t) = \int d\omega\ A(\omega) e^{[ik(\omega)z - i\omega t]} \qquad \omega = 2\pi f,\ k = \frac{2\pi}{\lambda}$$

- For wave packet near frequency $\omega_\circ$ (like spread-spectrum GPS signal), $k(\omega) \simeq k_\circ + \left.\frac{dk}{d\omega}\right|_{\omega=\omega_\circ} (\omega - \omega_\circ)$

- Then

$$u(z,t) = \underbrace{e^{i\left[k_\circ - \left.\frac{dk}{d\omega}\right|_{\omega=\omega_\circ} \omega_\circ\right] z}}_{\text{phase factor}} \underbrace{\int d\omega\ A(\omega) e^{i\omega\left[\left.\frac{dk}{d\omega}\right|_{\omega=\omega_\circ} z - t\right]}}_{u\left(0, \left.\frac{dk}{d\omega}\right|_{\omega=\omega_\circ} z - t\right)}$$

Group Velocity

$$u(z,t) \sim u\left(0, \left.\frac{dk}{d\omega}\right|_{\omega=\omega_\circ} z - t\right)$$

- Constant phase for $\left.\frac{dk}{d\omega}\right|_{\omega=\omega_\circ} z - t = \text{constant}$

- Thus, apart from overall phase factor, wave packet travels along undistorted in shape with **group velocity** $v_g = \left.\frac{d\omega}{dk}\right|_{k=c/\omega_\circ}$

Group and Phase Velocities

- Start with $k = \frac{n\omega}{c}$ where n is the (phase) refractive index

- Differentiate

  $$\frac{dk}{d\omega} = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_g}$$

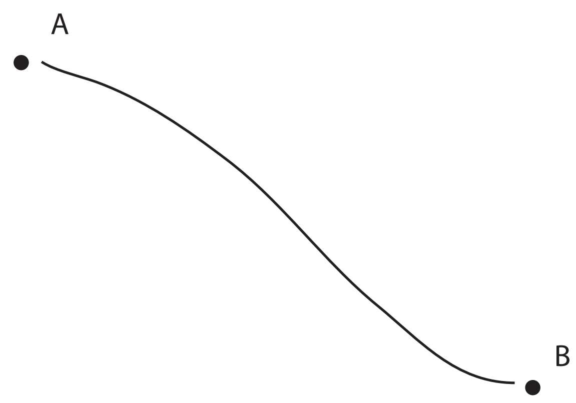
  $$v_g = v_p\left(1 + \frac{\omega}{n}\frac{dn}{d\omega}\right)^{-1} = \frac{c}{n_g}$$

- n_g is group refractive index

- If $\frac{dn}{d\omega} = 0$, $v_g = v_p$ and $n_g = n$

Propagation Delay

- Consider a radio signal propagating from point A to point B in a medium characterized by refractive index $n(\vec{x})$



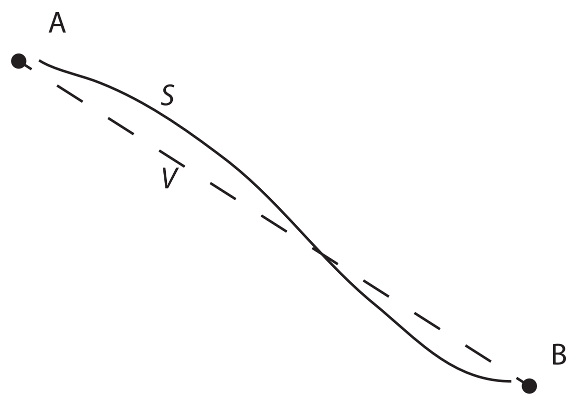
- The path is yet to be determined, but we know the path is a straight line for $n(\vec{x}) = 1$

Propagation Delay

- The speed of propagation is $\frac{c}{n}$, so the time of propagation along the path S is $\tau = \frac{1}{c}\int_S ds\; n(\vec{x})$

- We define the **propagation delay** as the difference between the propagation time along the path S in the medium and that for a fictitious signal propagating along the straight line in a vacuum

Propagation Delay



- Propagation delay $\Delta\tau$ in units of time:

$$\Delta\tau = \frac{1}{c}\int_S ds\; n(\vec{x}) - \frac{1}{c}\int_V ds$$

- Or in units of distance

$$\Delta\tau = \int_S ds\; n(\vec{x}) - \int_V ds$$

Propagation Delay

- Since $n \simeq 1$ for the atmosphere, we often write $n = 1 + 10^{-6}N$, where $N = 10^6(n-1)$ is the **refractivity**

- Then

$$\Delta\tau = \int_S ds\; n(\vec{x}) - \int_V ds = 10^{-6}\int_S ds\; N(\vec{x}) + \left[\int_S ds - \int_V ds\right]$$

- The term in brackets is due to the increased path length of the refracted signal

- The first term is the retarding of the signal along the signal path

The Ionosphere

- For the purposes of radio propagation at GPS frequencies (L-band), the atmosphere can be divided into two regimes

- The **ionosphere** is the part of the atmosphere consisting of “free” electrons weakly bound to charged atoms and molecules

- Typical altitudes above Earth’s surface 85–600 km

Ionospheric Refractive Index

- A classical model for the ionospheric refractive index starts with an electron bound by a harmonic force and acted on by an electric field $\vec{E}$

  $$-e\vec{E}(\vec{x},t) = m\left[\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_n^2\vec{x}\right]$$

- LHS: Force on electron with charge e acted on by $\vec{E}$

- RHS: (1) ma; (2) phenomenonological damping force; (3) restoring force with natural frequency ω_n

Ionospheric Refractive Index

- If $\vec{E}(\vec{x},t) = \vec{E}e^{i\omega t}$ the solution is

  $$\vec{x} = -\tfrac{e}{m}\left(\omega_n^2 - \omega^2 - i\omega\gamma\right)^{-1}\vec{E}$$

- The dipole moment formed by a single electron is

  $$\vec{p} = -e\vec{x} = \tfrac{e^2}{m}\left(\omega_n^2 - \omega^2 - i\omega\gamma\right)^{-1}\vec{E}$$

- If there are N_e dipoles (electrons) per unit volume

  $$\vec{P} = N_e\vec{p} = \tfrac{N_e e^2}{m}\left(\omega_n^2 - \omega^2 - i\omega\gamma\right)^{-1}\vec{E} = \chi\vec{E}$$

Ionospheric Refractive Index

- Thus the electric susceptibility χ is

$$\chi(\omega) = \frac{N_e e^2}{m}\left(\omega_n^2 - \omega^2 - i\omega\gamma\right)^{-1}$$

- The dielectric constant is

$$\epsilon(\omega) = 1 + 4\pi\chi(\omega) = 1 + \frac{4\pi N_e e^2}{m}\left(\omega_n^2 - \omega^2 - i\omega\gamma\right)^{-1}$$

- For radio waves it is found that $\omega \gg \omega_n$, so

$$\epsilon(\omega) \simeq 1 - 4\pi\chi(\omega) = 1 - \frac{4\pi N_e e^2}{m}\frac{1}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

- ω_p is called the **plasma frequency**

Ionospheric Refractive Index

- For a non-magnetic medium $\mu = 1$, so the refractive index is

  $n = \sqrt{\mu\epsilon} \simeq 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}$

- From relation between group and phase velocities the group refractive index is

  $n_g = n\left(1 + \frac{\omega}{n}\frac{dn}{d\omega}\right) = \left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)\left(1 + \frac{1}{n}\frac{\omega_p^2}{\omega^2}\right) \simeq 1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}$

- For GPS, we associate n with the carrier beat phase and n_g with the pseuodrange

Ionospheric Delay—Pseudorange

- The propagation delay for the pseudorange is

  $$\Delta\tau = 10^{-6}\int_S ds\; N_g(\vec{x}) + \left[\int_S ds - \int_V ds\right]$$

- For the ionosphere, bending can be shown to be small, i.e., $S = V$

- The group refractivity is

  $$N_g = 10^6(n_g - 1) \simeq 10^6 \times \frac{1}{2}\frac{\omega_p^2}{\omega^2} = 10^6 \times \frac{1}{2}\frac{f_p^2}{f^2}$$

  with $f_p^2 = \frac{N_e e^2}{\pi m}$

Ionospheric Delay—Pseudorange

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- For the ionosphere, bending can be shown to be small, i.e., $S = V$

- Then the ionospheric delay for the pseudorange (units of length) is

$$\Delta\tau_{\text{ion}} \simeq \int ds\ \frac{N_e e^2}{2\pi m f^2} = \frac{e^2}{2\pi m f^2}\int ds\ N_e(s)$$

Ionospheric Delay—Pseudorange

- The electron density N_e can vary by orders of magnitude through the ionosphere

- However, the ionospheric delay depends on the *integrated* electron density, called the **total electron content** (TEC)

  $$\text{TEC} = \int ds\ N_e(s)$$

- Ionospheric delay for the pseudorange (units of length)

  $$\Delta\tau_{\text{ion}} = \frac{e^2}{2\pi m f^2}\text{TEC} = \frac{40.3\ \text{m}^3\ \text{s}^{-2}}{f^2}\text{TEC}$$

Pseudorange: Dual-Frequency Ionospheric Correction

- Pseudorange observation equation for L_j frequency ($j = 1, 2$) including ionospheric delay

  $$R_j = \tilde{\rho} + \Delta\tau_{\text{ion}} + C = \tilde{\rho} + \frac{A}{f_j^2} + C$$

- $\tilde{\rho}$ is range corrected for satellite motion, C is combined clock

- Only second term depends on frequency

- Time, satellite, site indices left off

Pseudorange: Dual-Frequency Ionospheric Correction

- Combine L1 and L2 pseudorange observations as

$$R_1 - \left(\frac{f_2}{f_1}\right)^2 R_2 = \left[1 - \left(\frac{f_2}{f_1}\right)^2\right] (\tilde{\rho} + C)$$

- Define the LC (linear combination) pseudorange

$$R_{\mathsf{LC}} = \left[1 - \left(\frac{f_2}{f_1}\right)^2\right]^{-1} \left[R_1 - \left(\frac{f_2}{f_1}\right)^2 R_2\right]$$

- Then the model for R_{LC} has no ionosphere terms

$$R_{\mathsf{LC}} = \tilde{\rho} + C$$

Carrier Beat Phase: Dual-Frequency Ionospheric Correction

- The phase refractive index was $n \simeq 1 - \frac{1}{2}\frac{f_p^2}{f^2}$

- Compare to group refractive index was $n \simeq 1 + \frac{1}{2}\frac{f_p^2}{f^2}$

- Thus ionospheric phase delay (units of length) is negative

  $\Delta\tau_{\text{ion}} = -\frac{40.3\ \text{m}^3\ \text{s}^{-2}}{f^2}\text{TEC}$

Carrier Beat Phase: Dual-Frequency Ionospheric Correction

- With phase in cycles, observation model

  $\phi_j = \frac{1}{\lambda_j}(\tilde{\rho} + c\delta) + N_j - \frac{1}{\lambda_j}\frac{A}{f_j^2}$

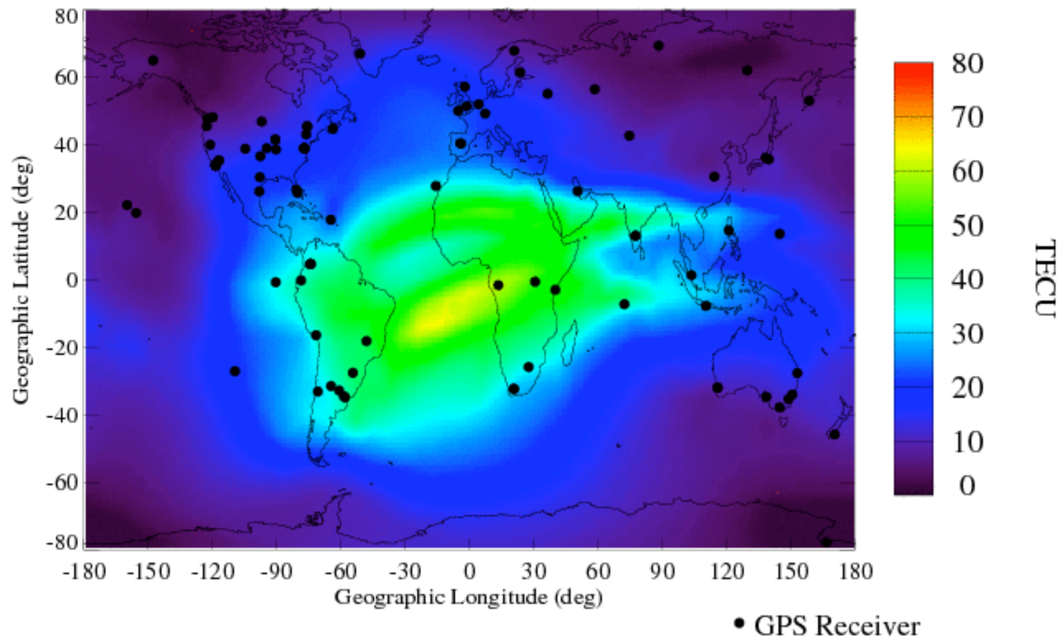
- We combine as $\phi_1 - \beta\phi_2$ with $\beta = f_2/f_1$

  $\phi_1 - \beta\phi_2 = \left(1 - \beta^2\right)\left(\frac{f_1}{c}\tilde{\rho} + \delta\right) + N_1 - \beta N_2$

- Integer ambiguities combine to create non-integer term

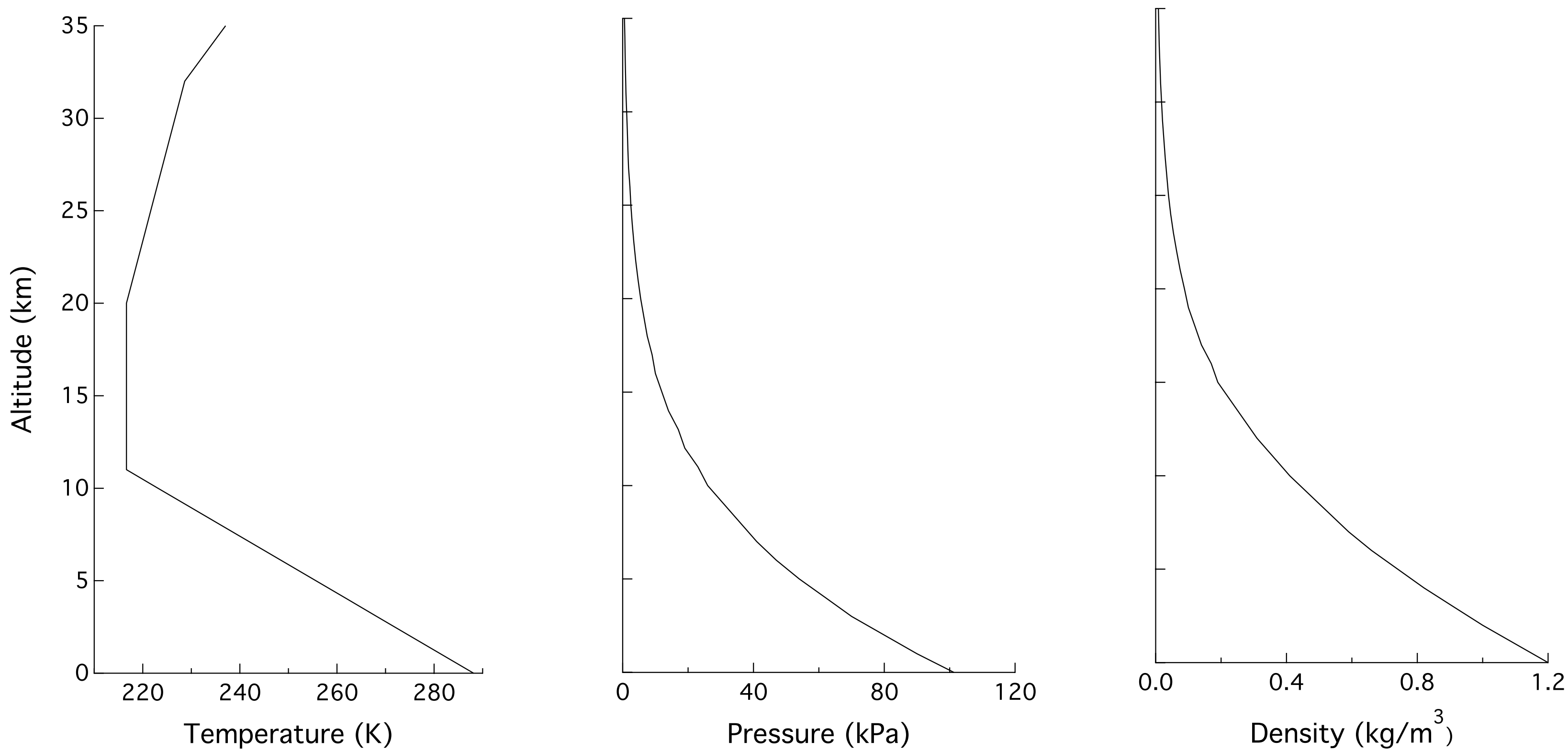
TEC Maps from GPS

- Can combine L1 and L2 to solve for TEC



- 1 TECU $= 10^{16}$ m$^{-2} \rightarrow \Delta\tau_{\text{ion}}^{\text{ph}}(\text{L1}) = -0.162$ m

Earth's Neutral Atmosphere



The Neutral Atmosphere

- The chemical composition of **dry air** is nitrogen (78.08%), oxygen (20.95%), argon (0.93%), and others at $< 1\%$ fractional volume

- Fractional volumes for dry air are very stable, except CO_2: 314 ppmv in 1960 to $\sim$385 ppmv today

- Water vapor is also highly variable in space and time, with relative humidities varying from 0% to 100%

- Atmospheric water vapor is located in troposphere

Atmospheric Refractivity

- Unlike the ionosphere the atmosphere is not dispersive below (say) 100 GHz

- None of the molecular constituents of dry air has a permanent dipole moment

- The induced dipole moment per unit volume scales with density

- The refractivity of dry air therefore is $N_d = A\rho_d$ with A being experimentally determined

Refractivity of H_2O(v): Permanent Dipole

- Water vapor has a permanent dipole moment

- However, if we think of water vapor as being a collection of randomly oriented dipoles, the dipole moment per unit volume will be zero

- Because the molecules are energetic, they are constantly moving and re-orienting, but except for very small statistical fluctuations the net dipole moment will be zero

Refractivity of H_2O(v): Permanent Dipole

- Under the influence of an applied electric field, the dipoles will be free to orient themselves, and create an net induced dipole moment

- But the electric field of a GPS signal is so weak, and the molecules so energetic, that there won't be a complete alignment

- Using a statistical mechanical argument, the probability of a molecule having an energy W is proportional to e^{-W/k_BT} where k_B is Boltzmann's constant and T is the absolute temperature

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Refractivity of H_2O(v): Permanent Dipole

- The potential energy of a permanent dipole $\vec{p}$ in a electric field $\vec{E}$ is

  $$W = -\vec{p} \cdot \vec{E} = -p_{\text{align}} E = -pE\cos\theta$$

- Here p_{align} is the component of the dipole moment aligned with the $\vec{E}$, and θ is the angle between the total dipole moment and $\vec{E}$

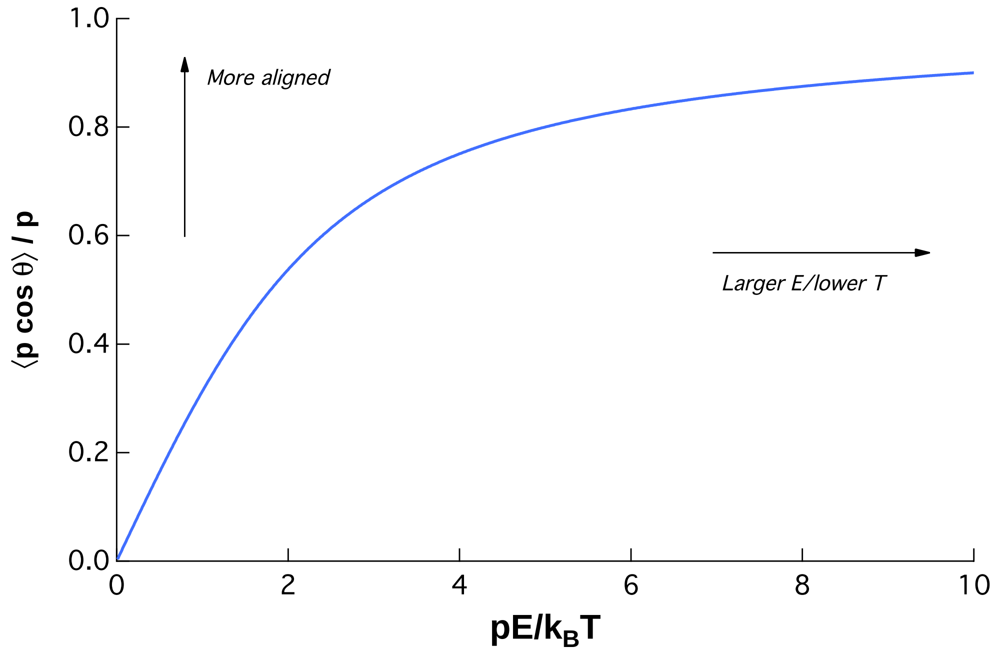
- The average value for p_{align} from an ensemble of dipoles using the probability density $e^{-W/k_B T}$ is

  $$\langle p\cos\theta \rangle = \frac{\int d\Omega\; p\cos\theta e^{+pE\cos\theta/k_B T}}{\int d\Omega\; e^{+pE\cos\theta/k_B T}}$$

Refractivity of H_2O(v): Permanent Dipole

- Doing the integration yields

$$\langle p \cos\theta \rangle = p \left[\coth\left(\frac{pE}{k_B T}\right) - \frac{k_B T}{pE} \right]$$



Refractivity of H_2O(v): Permanent Dipole

- For atmospheric water vapor $pE/k_BT \ll 1$

- We expand $\coth x \simeq \frac{1}{x} + \frac{1}{3}x$ to get

  $\langle p \cos\theta \rangle \simeq \frac{p^2 E}{3k_BT}$

- For number density N_w of H_2O(v) molecules, the polarization is $P = N_w p^2 E/k_BT = \chi E$

- Where recall χ is the susceptibility, $\epsilon = 1 + 4\pi\chi$, and refractive index $n \simeq 1 + 2\pi\chi$

Refractivity of Moist Air

- There is also an induced dipole part, so the refractivity $N = 10^6(n-1)$ of water vapor is

  $N = B\rho_w + C\frac{\rho_w}{T}$

- First term is induced dipole moment, second is permanent

- The total radio refractivity of moist air is thus

  $N = A\rho_d + B\rho_w + C\frac{\rho_w}{T^2}$

Refractivity of Moist Air

- The canonical way of writing the refractivity (using the ideal gas law $P = \rho RT$) is

  $N = k_1 \frac{p_d}{T} + k_2 \frac{p_w}{T} + k_3 \frac{p_w}{T^2}$

- p_d is partial pressure of dry gases, p_w is partial pressure of w.v.

- The constants have been measured experimentally: $k_1 \simeq 77.67 \pm 0.01$K/hPa, $k_2 \simeq 72 \pm 10$ K/hPa, and $k_3 \simeq (3.75 \pm 0.03) \times 10^5$ K^2/hPa

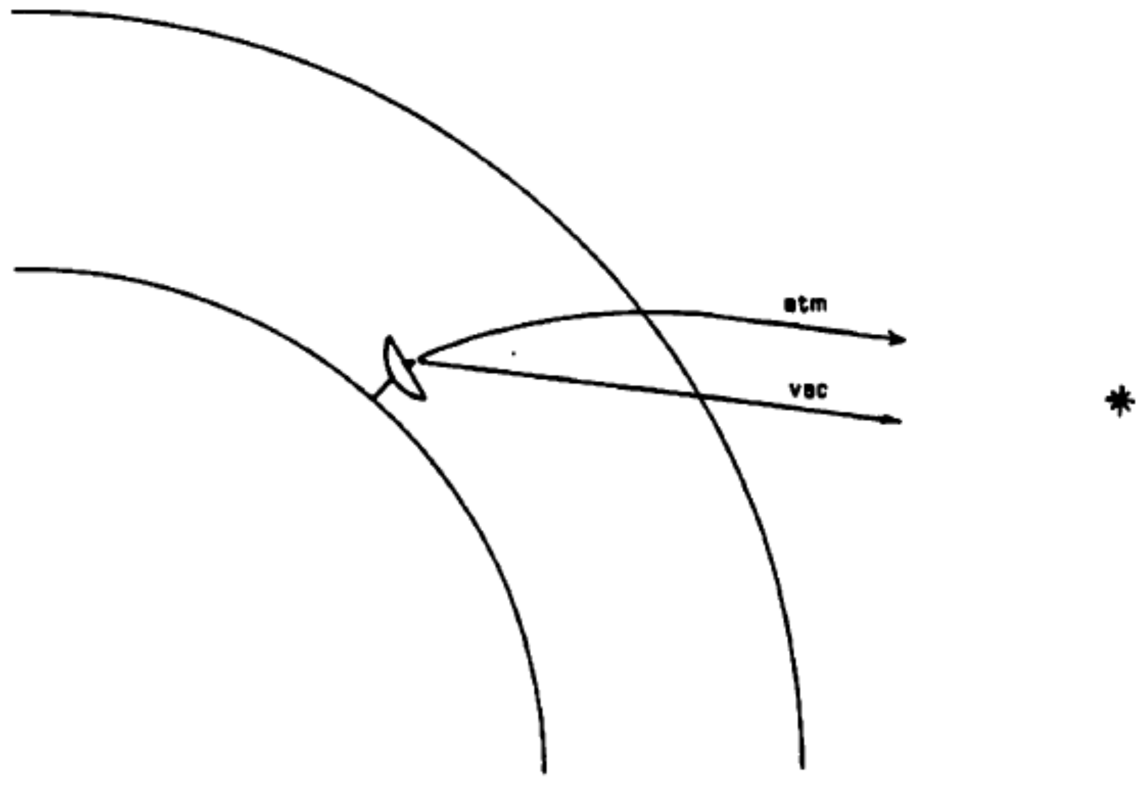
Atmospheric Propagation Delay

- Recall the expression for the propagation delay:

  $$\Delta\tau = 10^{-6}\int_S ds\; N(\vec{x}) + \left[\int_S ds - \int_V ds\right]$$

- The term in brackets represents the geometric difference between the length of the refracted and hypothetical (*in vaccuo*) unrefracted ray paths

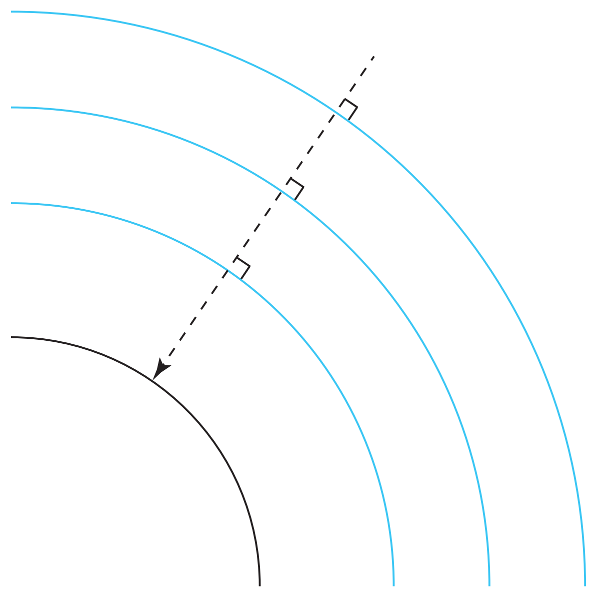
Atmospheric Propagation Delay



Atmospheric Propagation Delay

- Consider spherically Earth, stratified atmosphere

- For observation from surface in zenith direction, ray travels normal to the layers with no bending



Zenith Propagation Delay

- This is called the **zenith delay** $\Delta\tau^z = 10^{-6}\int_0^\infty dz\, N(z)$

- Using the expression for the refractivity
  $N = k_1\frac{P}{T} + k_2\frac{p_w}{T} + k_3\frac{p_w}{T^2}$

- First two terms can be written using $P = \rho RT$ as
  $k_1 R_d \rho_d + k_2 R_w \rho_w = k_1\rho + k_2'\frac{p_w}{T}$
  where ρ is total density and $k_2' = k_2 - k_1(M_w/M_d)$

- M's are molar masses, R's are specific gas constants

Zenith Hydrostatic Delay

- The contribution of the first term to the zenith delay is $10^{-6} k_1 R_d \int_0^\infty dz\ \rho(z)$

- For atmosphere in hydrostatic equilibrium $dz\,\rho = -dP/g(z)$ and the **hydrostatic delay** is

  $$\Delta\tau_h^z = 10^{-6} k_1 R_d P_\circ / g_m \simeq (2.2768\ \text{mm/hPa}) P_\circ$$

- $P_\circ$ is surface pressure; mean gravity $g_m \simeq 9.784\ \text{m/s}^2$ at sea level

- For $P_\circ = 1013$ hPa, $\Delta\tau_h^z = 2.3064$ m

Zenith Wet Delay

- The zenith delay forumla is

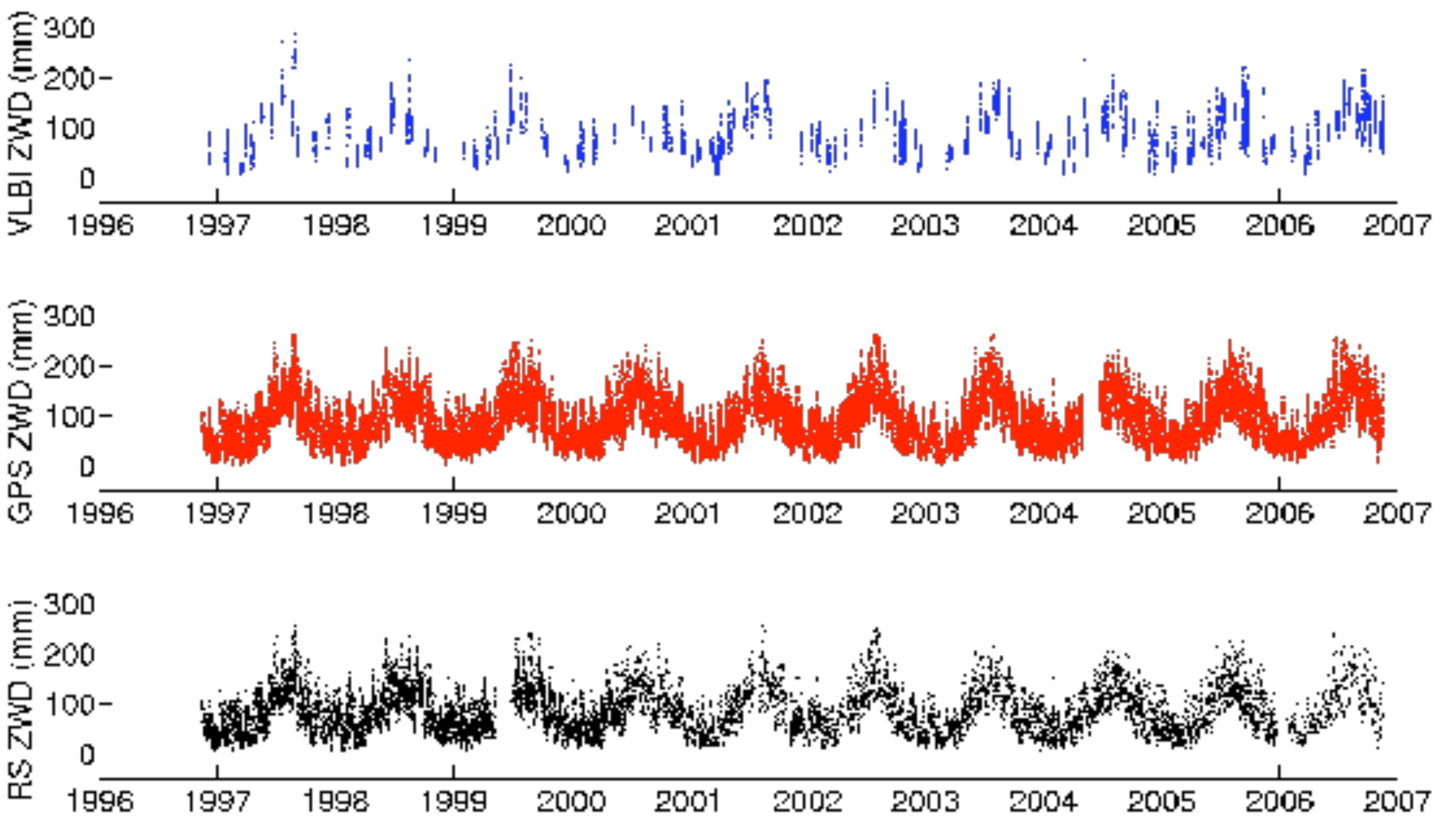
  $$\Delta\tau^z = \Delta\tau_h^z + 10^{-6}\int_0^\infty dz\, \frac{p_w}{T}\left[k_2' + \frac{k_3}{T}\right]$$

- Second term depends only on water vapor (not dry constituents), is known as the **zenith wet delay**

- The zenith wet delay ranges from $\sim$0 to $\sim$40 cm and is highly variable in time and space because water vapor is

- It's also hard to model to required accuracy

Zenith Wet Delay



Zenith wet delay measured from three techniques at Onsala (Sweden) Space Observatory

Atmospheric Mapping Function

- What about off-zenith directions?

- For a flat Earth with a homogeneous atmosphere, we have **secant law**: $\Delta\tau(\epsilon) = \Delta\tau^z \csc\epsilon$

- ϵ is elevation angle (angle above horizon)

- In analogy with cosecant law, we introduce the **mapping function** $m(\epsilon)$: $\Delta\tau(\epsilon) = \Delta\tau^z m(\epsilon)$

Atmospheric Mapping Function

- For a spherically layered atmosphere, $m(\epsilon)$ can be approximated by a continued fraction

$$m(\epsilon) = \cfrac{A}{\sin\epsilon + \cfrac{a_1}{\sin\epsilon + \cfrac{a_2}{\sin\epsilon + \cdots}}}$$

- Coefficients (usually 2–3) determined using ray-tracing

- A ($\simeq 1$) depends on a_i since $m(90^\circ) = 1$

- One of the latest mapping functions ray-traces through the daily ECMWF weather models and determines coefficients on a $2.5^\circ \times 2.0^\circ$ grid

Estimation of the Atmospheric Delay

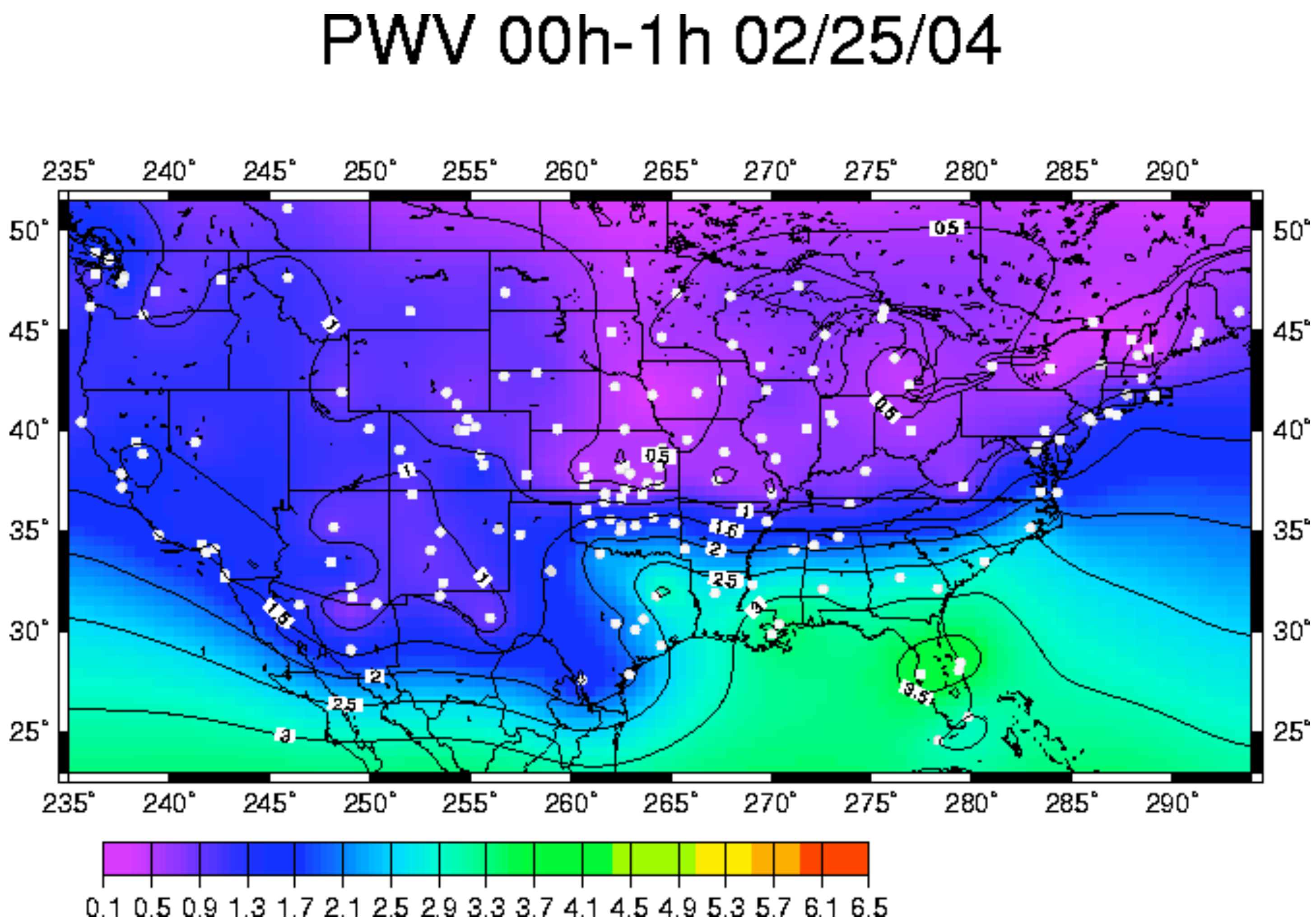
- Phase model (cycles)

  $$\phi_j = \frac{1}{\lambda_j}(\tilde{\rho} + c\delta) + N_j - \frac{1}{\lambda_j}\frac{A}{f_j^2} + \frac{1}{\lambda_j}\Delta\tau^z m(\epsilon)$$

- Wet zenith delay is unknown and hard to model

- We could estimate using $m(\epsilon)$ as partial derivative

- Like ionosphere, this “noise” is someone else’s signal



Zenith precipitable water vapor ($\simeq \Delta\tau^z/6.7$) from ground-based GPS observations [UCAR]