EESC 9945

Geodesy with the Global Positioning System

Class 7: *Relative Positioning using Carrier-Beat Phase*

GPS Carrier Phase

- The model for the carrier-beat phase observable for receiver p and satellite r we developed, ignoring atmospheric and ionospheric propagation delays, was

$$\phi_p^r(t) = -\frac{1}{\lambda}\rho_p^r(t) + \phi_\circ^r - \phi_p^\circ + N_p^r + f_\circ[\delta^r(t) - \delta_p(t)]$$

- We've left off the "Δ" since too many

- Subscript letters are different

Single Difference

- The **single difference** (or "single difference phase") is the difference between phase observations from two sites for a single satellite:

$$
\begin{aligned}
\phi^r_{pq}(t) &= \phi^r_p(t) - \phi^r_q(t) \\
&= -\frac{1}{\lambda}\left[\rho^r_p(t) - \rho^r_q(t)\right] - \phi^\circ_p + \phi^\circ_q + N^r_p - N^r_q \\
&\quad\; + f_\circ[\delta_q(t) - \delta_p(t)]
\end{aligned}
$$

- Satellite initial phase and clock error have differenced out

Baseline Vector

- Let $\vec{b}_{pq}$ be the vector between two GPS site located at $\vec{x}_p$ and $\vec{x}_q$:

  $\vec{b}_{pq} = \vec{x}_q - \vec{x}_p$

- $\vec{b}_{pq}$ is called the **baseline vector** (or "inter site vector") between sites p and q

Single-difference geometry: Short baseline limit

- The geometry term in the single-difference phase is

$$\Delta\rho_{pq}^r = \rho_p^r - \rho_q^r = |\vec{x}^r - \vec{x}_p| - |\vec{x}^r - \vec{x}_q|$$

- Substituting $\vec{x}_q = \vec{x}_p + \vec{b}_{pq}$ we have

$$\Delta\rho_{pq}^r = \rho_p^r - \left|\vec{x}^r - \vec{x}_p - \vec{b}_{pq}\right| = \rho_p^r - \left|\vec{\rho}_p^r - \vec{b}_{pq}\right|$$

- The length of the vector $\vec{\rho}_p^r - \vec{b}_{pq}$ is

$$\left|\vec{\rho}_p^r - \vec{b}_{pq}\right| = \rho_p^r \left[1 - 2\frac{\vec{\rho}_p^r \cdot \vec{b}_{pq}}{\rho_p^r} + \left(\frac{b_{pq}}{\rho_p^r}\right)^2\right]^{1/2}$$

Single-difference geometry: Short b limit

- Expanding for small $|b_{pq}|$ gives

$$\left|\vec{\rho}_p^{\,r} - \vec{b}_{pq}\right| \simeq \rho_p^r \left[1 - \frac{\widehat{\rho}_p^{\,r} \cdot \vec{b}_{pq}}{\rho_p^r} + o\left(\left(\frac{b_{pq}}{\rho_p^r}\right)^2\right)\right]$$

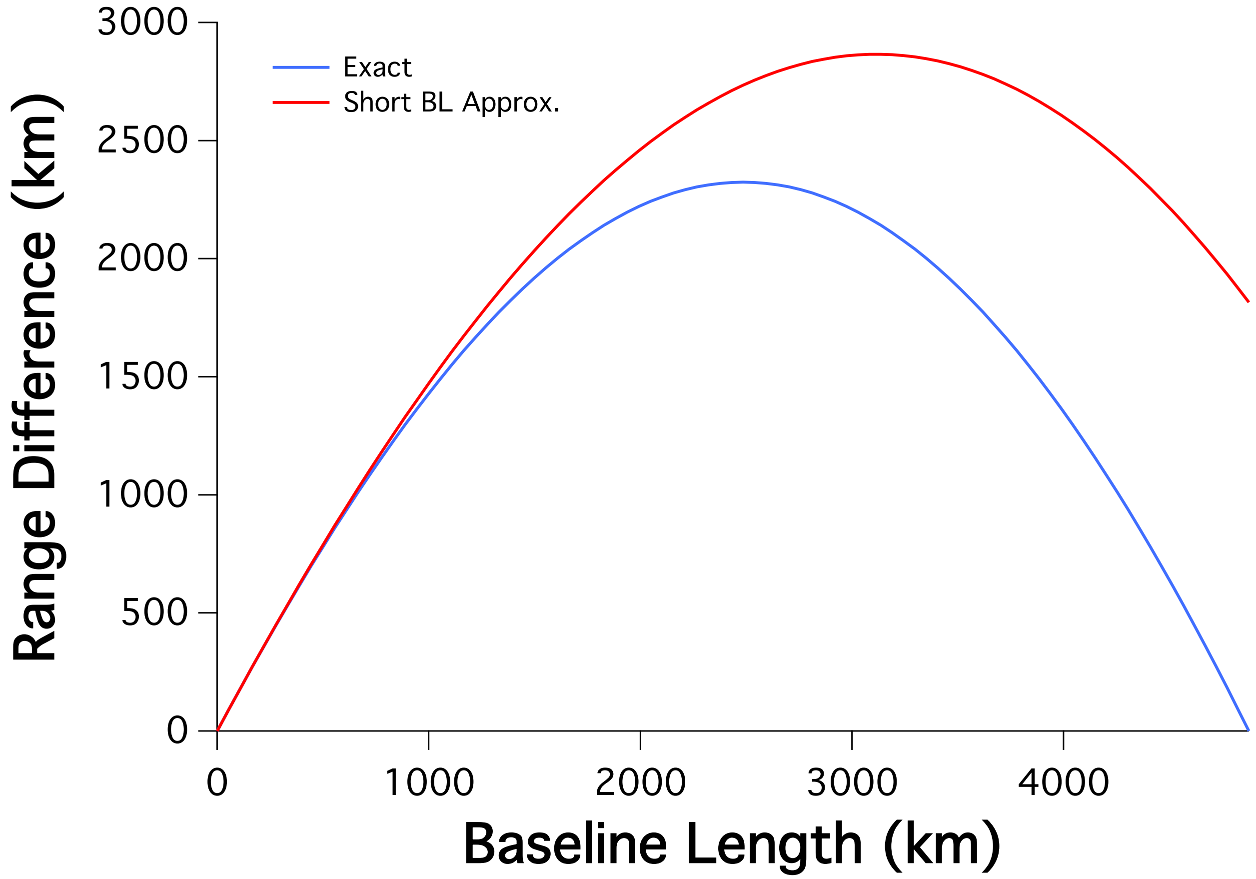
- Dropping terms of order $(b_{pq}/\rho_p^r)^2$ and smaller gives

$$\left|\vec{\rho}_p^{\,r} - \vec{b}_{pq}\right| \simeq \rho_p^r - \widehat{\rho}_p^{\,r} \cdot \vec{b}_{pq}$$

- The geometry term in the single-difference phase for $b_{pq} \ll \rho_p^r$ is thus

$$\Delta\rho_{pq}^r = \rho_p^r - \left|\vec{\rho}_p^{\,r} - \vec{b}_{pq}\right| \simeq \widehat{\rho}_p^{\,r} \cdot \vec{b}_{pq}$$

Single-difference geometry: Short b limit



Difference is $\sim$10 cm @ 100 km

Implications of differencing

- Single-difference phase observations are much less sensitive to position and much more sensitive to difference of position

- In other words, sensitivity to coordinate origin has been significantly decreased

Implications of differencing

- Sensitivity of phase to baseline error (as rule of thumb) is

  $\delta\Phi \simeq \delta b + b\delta\widehat{\rho} \simeq \delta b + b\frac{\delta x_{\mathsf{sat}}}{x_{\mathsf{sat}}}$

- Thus, an error in the orbital position of a GPS satellite can compensated (i.e., $\Delta\Phi \simeq 0$) if

  $\left|\frac{\delta b}{b}\right| \simeq \left|\frac{\delta x_{\mathsf{sat}}}{x_{\mathsf{sat}}}\right|$

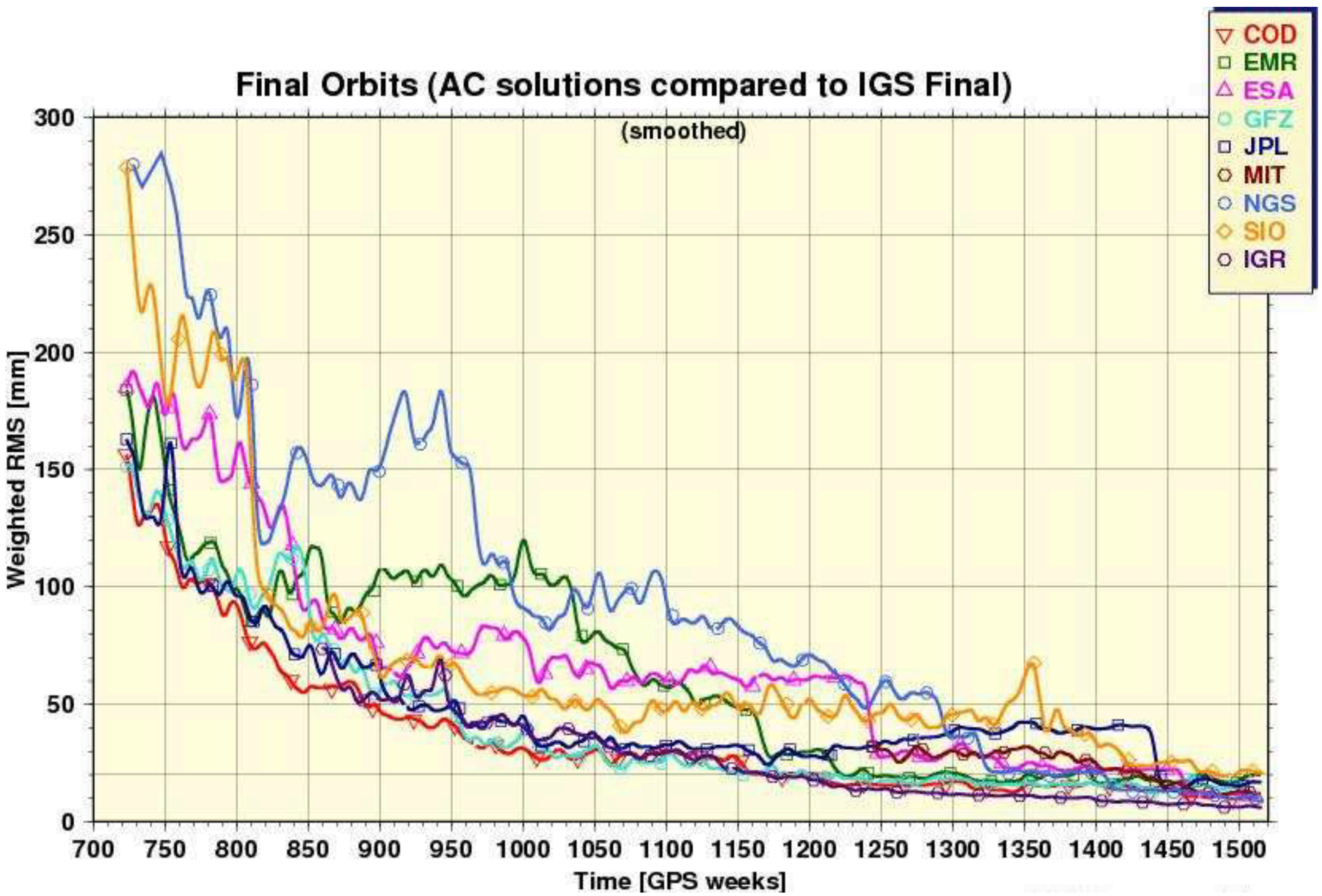
- This is why orbit errors are often spoken of in terms of fractional errors (e.g., parts-per-billion)

Errors in IGS Orbit Products

IGS Product Name	Latency	Nominal Accuracy (cm)	Nominal Accuracy* (ppb)
Broadcast	0	100	40
Ultra-Rapid	0–9 h	3–5	0.1–0.2
Rapid	17–41 h	2.5	0.09
Final	12–18 d	2.5	0.09

1 ppb orbit error $\rightarrow$ $\sim$1 mm baseline error @ 1000 km

Error History for IGS Orbit Products



Double Difference

- Single difference for sites p and q to satellite r:

$$\begin{aligned}\phi_{pq}^r(t) &= \phi_p^r(t) - \phi_q^r(t) \\ &= -\frac{1}{\lambda}\left[\rho_p^r(t) - \rho_q^r(t)\right] - \phi_p^\circ + \phi_q^\circ + N_p^r - N_q^r \\ &\quad + f_\circ[\delta_q(t) - \delta_p(t)]\end{aligned}$$

- **Double difference** formed from two single differences to two satellites:

$$\begin{aligned}\phi_{pq}^{rs}(t) &= \phi_{pq}^r(t) - \phi_{pq}^s(t) \\ &= -\frac{1}{\lambda}\left[\rho_p^r(t) - \rho_q^r(t) - \rho_p^s(t) + \rho_q^s(t)\right] + N_p^r - N_q^r - N_p^s + N_q^s \\ &= -\frac{1}{\lambda}\rho_{pq}^{rs}(t) + N_{pq}^{rs}\end{aligned}$$

- All site-only- and satellite-only-dependent terms have canceled

Working with Double Differences

- Some advantages:
  1. No clocks, initial phases
  2. Simple model
- Some disadvantages:
  1. Bookkeeping
  2. Estimating ambiguities
  3. Further loss of sensitivity
  4. Unable to reconstruct “one-way” phases

Triple Difference

- Double difference formed from two single differences to two satellites:

$$\begin{aligned} \phi_{pq}^{rs}(t) &= \phi_{pq}^{r}(t) - \phi_{pq}^{s}(t) \\ &= -\frac{1}{\lambda}\left[\rho_p^r(t) - \rho_q^r(t) - \rho_p^s(t) + \rho_q^s(t)\right] + N_p^r - N_q^r - N_p^s + N_q^s \\ &= -\frac{1}{\lambda}\rho_{pq}^{rs}(t) + N_{pq}^{rs} \end{aligned}$$

- **Triple difference** formed from time differences of double difference

$$\begin{aligned} \Delta\phi_{pq}^{rs}(t_j, t_k) &= \phi_{pq}^{rs}(t_k) - \phi_{pq}^{rs}(t_j) \\ &\simeq -\frac{1}{\lambda}\left[\dot\rho_p^r(t) - \dot\rho_q^r(t) - \dot\rho_p^s(t) + \dot\rho_q^s(t)\right](t_k - t_j) \end{aligned}$$

- Useful for cycle-slip detection

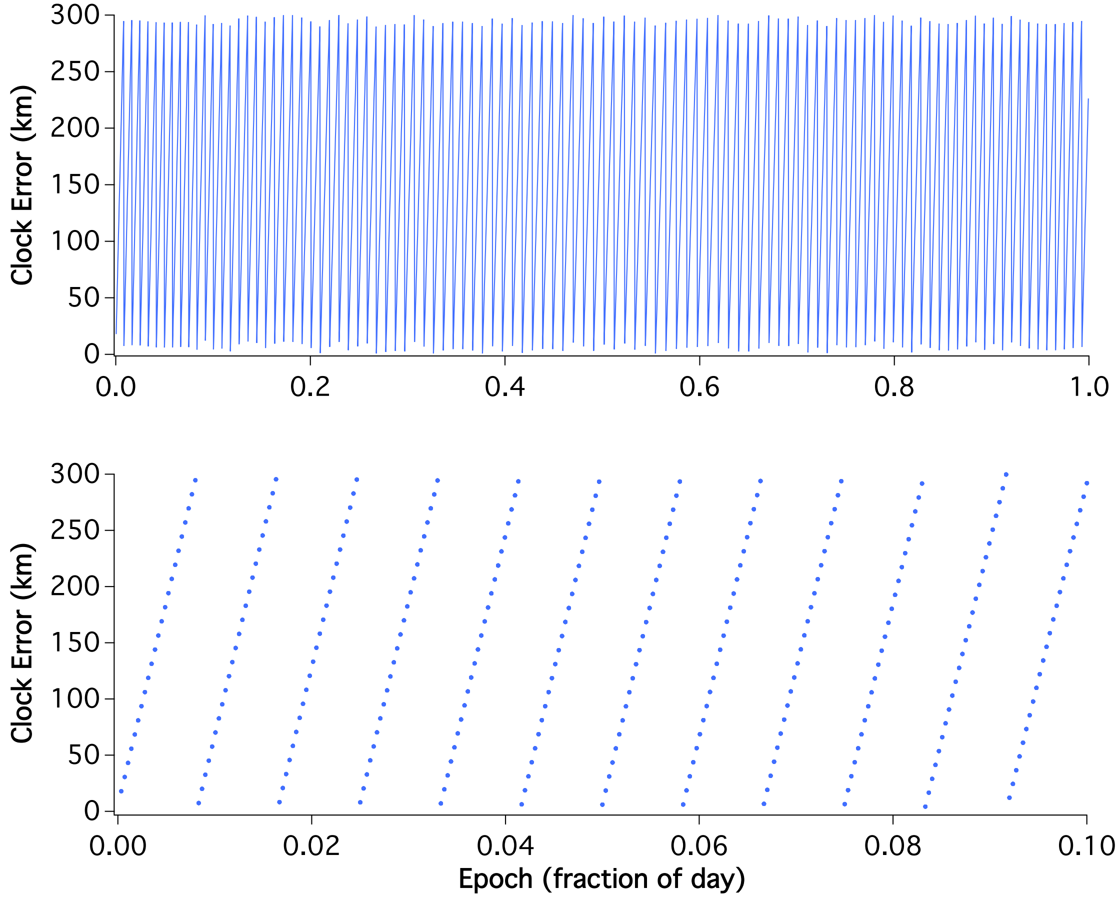
GPS Data Analysis: Outline

1. Perform “clock” solution

    (a) Estimate receiver clock errors

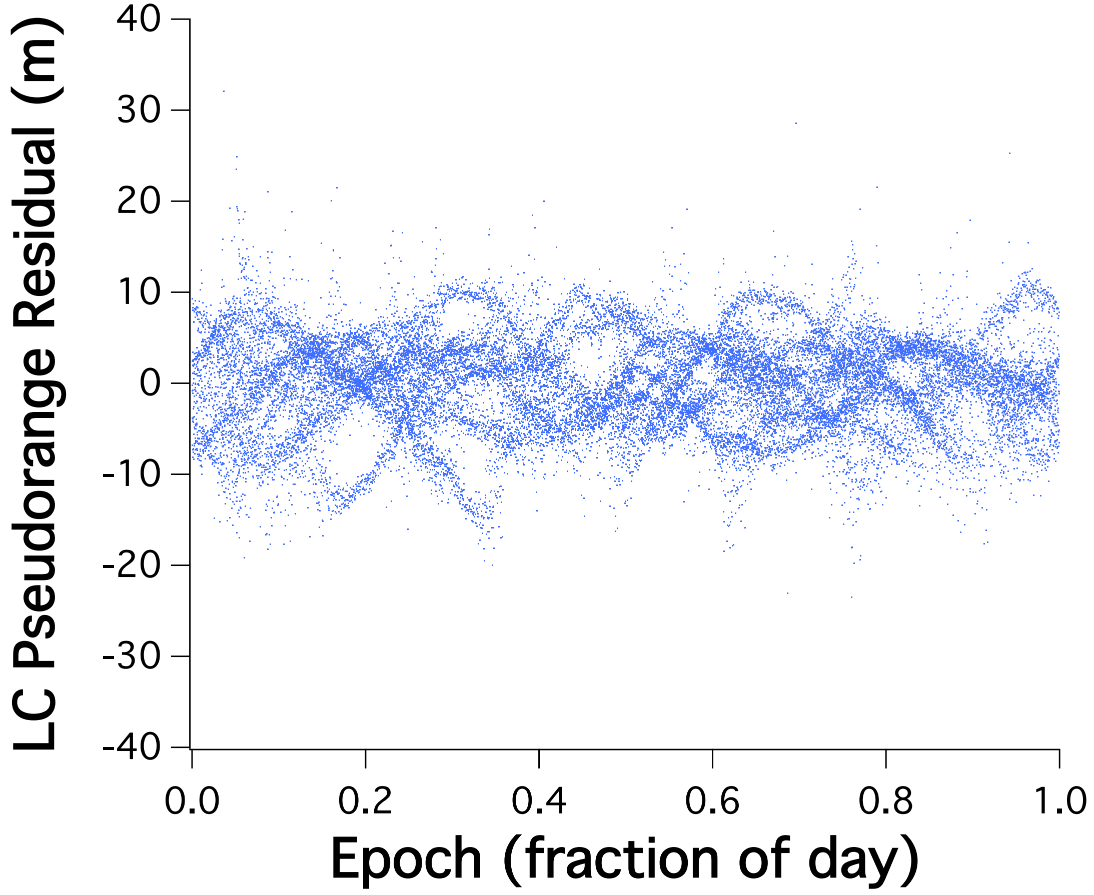
    (b) Estimate site position if no prior better (< 1 m) is available

Clock solution showing "clock steering"



300 km $\Longleftrightarrow$ 1 msec

Pseudorange Residuals



RMS = 5.4 m

GPS Data Analysis: Outline

1. Perform “clock” solution

   (a) Estimate receiver clock errors

   (b) Estimate site position if no prior better (<1 m) is available

2. Using best model, repair cycle slips

Data Processing: Cycle Slips

- We have assumed that the integer bias N in the phase model is not a function of time

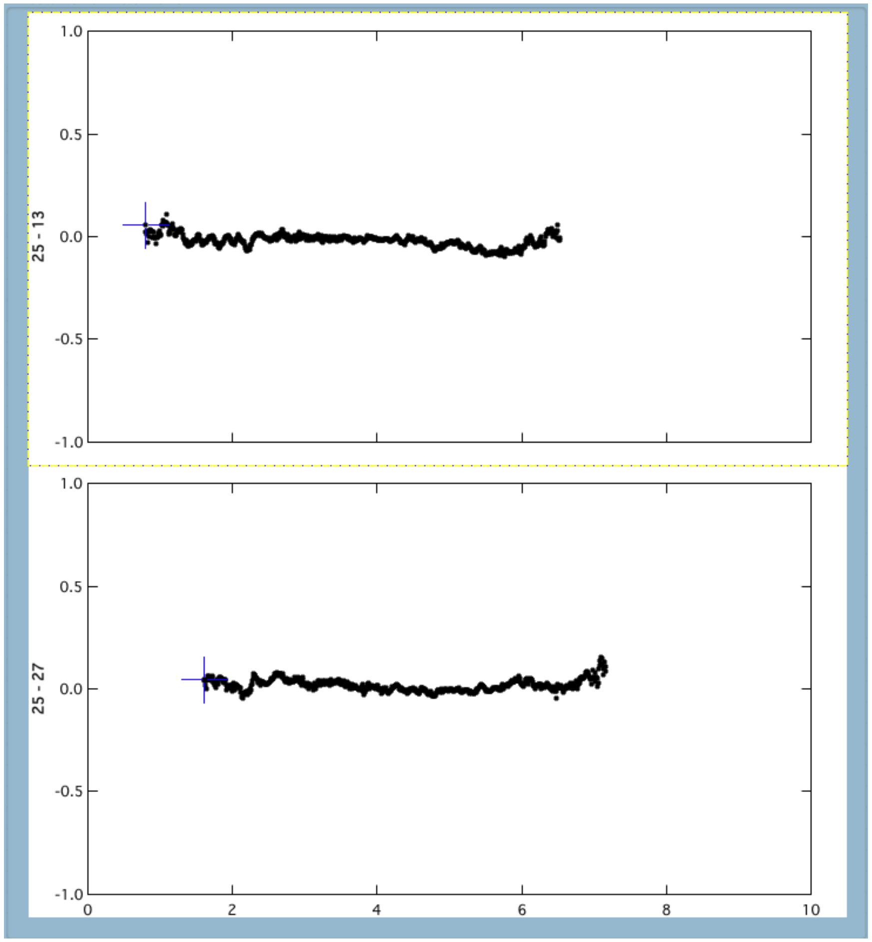
- In fact, "cycles slips" can occur in systems—such as GPS receivers—that employ phase-locked loops

- A phase-locked loop is a feedback system that uses the phase difference of the received and internal signals to modify the internal frequency in an attempt to match frequency variations of the incoming signal and to keep the signal "locked"

- Such systems are highly sensitive to SNR to be able to separate the signal (which should be locked onto) from the noise (which should be ignored)

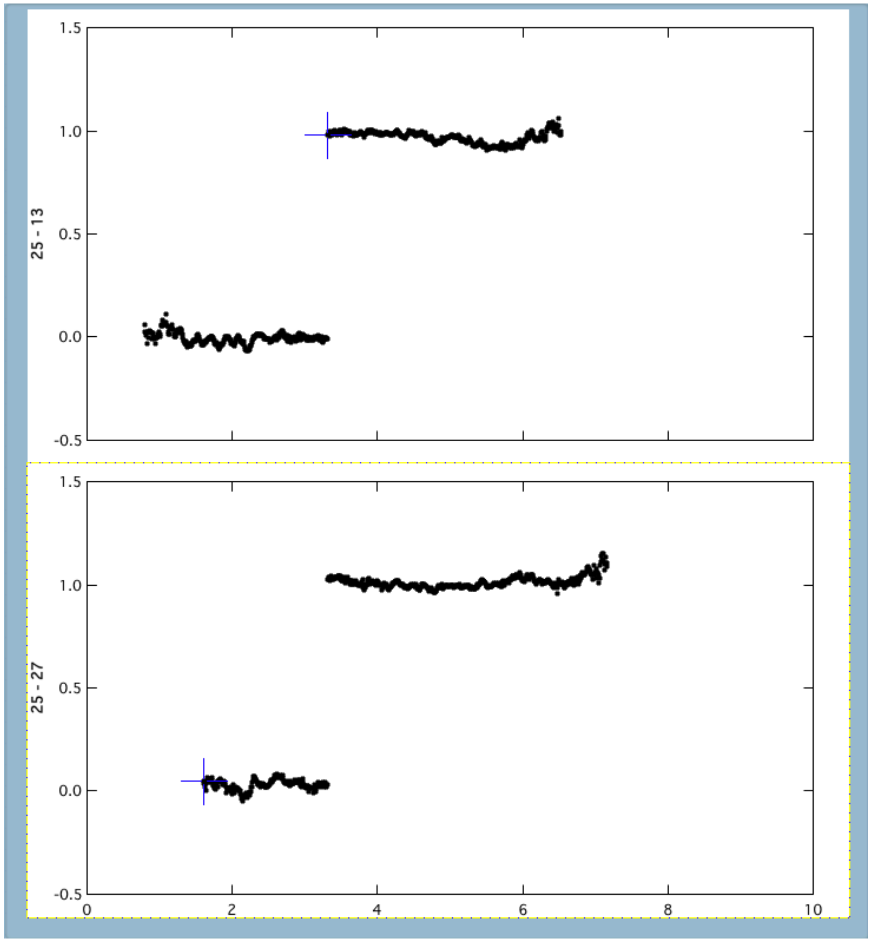
- Also, if the phase varies rapidly in a noise-like manner (e.g., ionosphere, multipath), the receiver can lose lock

L1 DD Residuals ($b = 10$ km)



Units: cycles/hours

L1 DD Residuals ($b = 10$ km) with cycle slip



Units: cycles/hours

Rectifying cycle slips

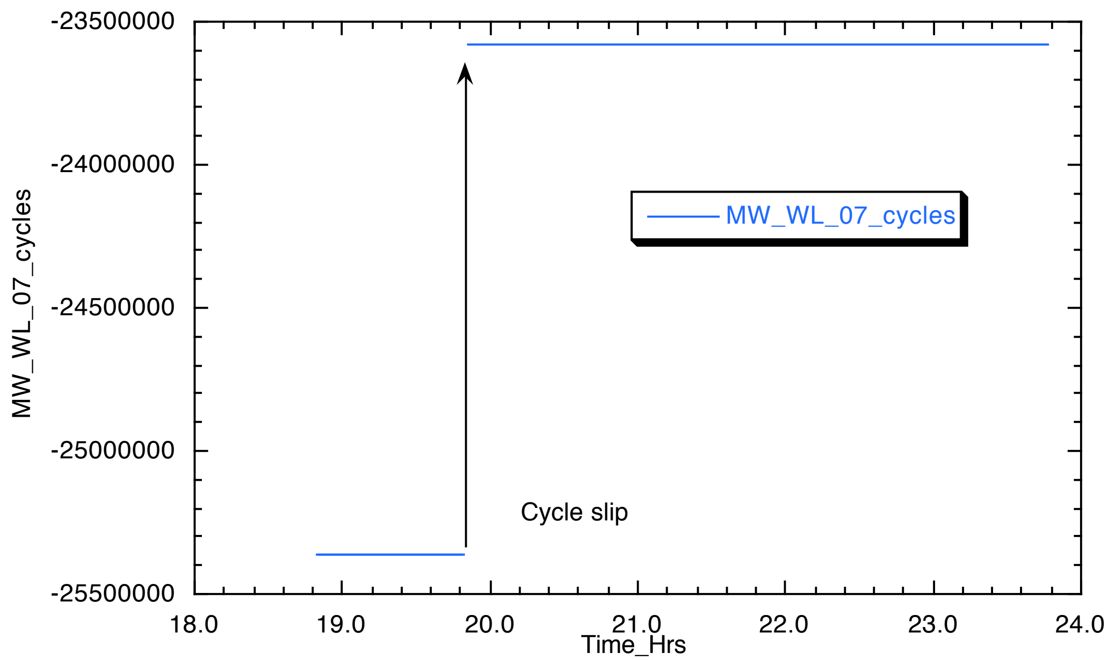
- These days, cycle slip detection and removal is done using automatic processing

- In old days, done by hand

- One useful data combination is the **Melbourne-Wübbena wide lane**:

$$\Delta_{\mathrm{mw}} = \phi_1 - \phi_2 - \frac{1}{c}\frac{(f_1 - f_2)}{(f_1 + f_2)}\left[f_1 R_1 + f_2 R_2\right] \qquad (1)$$

Melbourne-Wübbena Wide-Lane



From Herring, *Principles of the Global Positioning System*

Next time

- Regional solutions using fixed orbits
- Orbit integration
- Orbit parameter estimation
- The IGS network