Solution

(1) The vectors $cv = (c, 0)$ with whole numbers c are equally spaced points along the x axis (the direction of v). They include $(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)$. Adding all vectors $dw = (0, d)$ puts a full line in the y direction through those points. We have infinitely many *parallel lines* from $cv + dw =$ (*whole number, any number*). These are vertical lines in the xy plane, through equally spaced points on the x axis.

(2) The vectors cv with $c \geq 0$ fill a "half-line". It is the *positive* x axis, starting at $(0, 0)$ where $c = 0$. It includes $(\pi, 0)$ but not $(-\pi, 0)$. Adding all vectors dw puts a full line in the y direction crossing every point on that half-line. Now we have a *half-plane*. It is the right half of the xy plane, where $x \geq 0$.

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

1 Describe geometrically (as a line, plane, ...) all linear combinations of

(a) $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}4\\4\\4\end{bmatrix}$ (b) $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ (c) $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$

2 Draw the vectors $v = \begin{bmatrix}4\\1\end{bmatrix}$ and $w = \begin{bmatrix}-2\\2\end{bmatrix}$ and $v + w$ and $v - w$ in a single xy plane.

3 If $v + w = \begin{bmatrix}3\\1\end{bmatrix}$ and $v - w = \begin{bmatrix}1\\3\end{bmatrix}$, compute and draw v and w.

4 From $v = \begin{bmatrix}2\\1\end{bmatrix}$ and $w = \begin{bmatrix}1\\2\end{bmatrix}$, find the components of $3v + w$ and $v - 3w$ and $cv + dw$.

5 Compute $u + v$ and $u + v + w$ and $2u + 2v + w$ when

$$u = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad v = \begin{bmatrix}-3\\1\\-2\end{bmatrix}, \quad w = \begin{bmatrix}2\\-3\\-1\end{bmatrix}.$$

6 Every combination of $v = (1, -2, 1)$ and $w = (0, 1, -1)$ has components that add to ____. Find c and d so that $cv + dw = (4, 2, -6)$.

7 In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix}3\\1\end{bmatrix} + d\begin{bmatrix}0\\1\end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

WORKED EXAMPLES

1.2 A For the vectors $v = (3, 4)$ and $w = (4, 3)$ test the Schwarz inequality on $v \cdot w$ and the triangle inequality on $\|v + w\|$. Find $\cos\theta$ for the angle between v and w. When will we have *equality* $|v \cdot w| = \|v\|\,\|w\|$ and $\|v + w\| = \|v\| + \|w\|$?

Solution The dot product is $v \cdot w = (3)(4) + (4)(3) = 24$. The length of v is $\|v\| = \sqrt{9 + 16} = 5$ and also $\|w\| = 5$. The sum $v + w = (7, 7)$ has length $\|v + w\| = 7\sqrt{2} \approx 9.9$.

Schwarz inequality	$\lvert v \cdot w\rvert \le \|v\|\,\|w\|$ is	$24 < 25$.
Triangle inequality	$\|v + w\| \le \|v\| + \|w\|$ is	$7\sqrt{2} < 10$.
Cosine of angle	$\cos\theta = \frac{24}{25}$	(Thin angle!)

If one vector is a multiple of the other as in $w = -2v$, then the angle is 0° or 180° and $|\cos\theta| = 1$ and $|v \cdot w|$ *equals* $\|v\|\,\|w\|$. If the angle is 0°, as in $w = 2v$, then $\|v + w\| = \|v\| + \|w\|$. The triangle is flat.

1.2 B Find a unit vector u in the direction of $v = (3, 4)$. Find a unit vector U perpendicular to u. How many possibilities for U?

Solution For a unit vector u, divide v by its length $\|v\| = 5$. For a perpendicular vector V we can choose $(-4, 3)$ since the dot product $v \cdot V$ is $(3)(-4) + (4)(3) = 0$. For a *unit* vector U, divide V by its length $\|V\|$:

$$u = \frac{v}{\|v\|} = \frac{(3,4)}{5} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad U = \frac{V}{\|V\|} = \frac{(-4,3)}{5} = \left(-\frac{4}{5}, \frac{3}{5}\right)$$

The only other perpendicular unit vector would be $-U = \left(\frac{4}{5}, -\frac{3}{5}\right)$.

Problem Set 1.2

1 Calculate the dot products $u \cdot v$ and $u \cdot w$ and $v \cdot w$ and $w \cdot v$:

$$u = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \qquad v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad w = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

2 Compute the lengths $\|u\|$ and $\|v\|$ and $\|w\|$ of those vectors. Check the Schwarz inequalities $|u \cdot v| \le \|u\|\,\|v\|$ and $|v \cdot w| \le \|v\|\,\|w\|$.

3 Find unit vectors in the directions of v and w in Problem 1, and the cosine of the angle θ. Choose vectors that make 0°, 90°, and 180° angles with w.

4 Find unit vectors u_1 and u_2 in the directions of $v = (3, 1)$ and $w = (2, 1, 2)$. Find unit vectors U_1 and U_2 that are perpendicular to u_1 and u_2.

5 For any *unit* vectors v and w, find the dot products (actual numbers) of

(a) v and $-v$ (b) $v+w$ and $v-w$ (c) $v-2w$ and $v+2w$

6 Find the angle θ (from its cosine) between

(a) $v = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $v = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(c) $v = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $w = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ (d) $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

7 (a) Describe every vector $w = (w_1, w_2)$ that is perpendicular to $v = (2, -1)$.
(b) The vectors that are perpendicular to $V = (1, 1, 1)$ lie on a _____.
(c) The vectors that are perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a _____.

8 True or false (give a reason if true or a counterexample if false):

(a) If u is perpendicular (in three dimensions) to v and w, then v and w are parallel.
(b) If u is perpendicular to v and w, then u is perpendicular to $v + 2w$.
(c) If u and v are perpendicular unit vectors then $\|u - v\| = \sqrt{2}$.

9 The slopes of the arrows from $(0, 0)$ to (v_1, v_2) and (w_1, w_2) are v_2/v_1 and w_2/w_1. If the product v_2w_2/v_1w_1 of those slopes is -1, show that $v \cdot w = 0$ and the vectors are perpendicular.

10 Draw arrows from $(0, 0)$ to the points $v = (1, 2)$ and $w = (-2, 1)$. Multiply their slopes. That answer is a signal that $v \cdot w = 0$ and the arrows are _____.

11 If $v \cdot w$ is negative, what does this say about the angle between v and w? Draw a 2-dimensional vector v (an arrow), and show where to find all w's with $v \cdot w < 0$.

12 With $v = (1, 1)$ and $w = (1, 5)$ choose a number c so that $w - cv$ is perpendicular to v. Then find the formula that gives this number c for any nonzero v and w.

13 Find two vectors v and w that are perpendicular to $(1, 0, 1)$ and to each other.

14 Find three vectors u, v, w that are perpendicular to $(1, 1, 1, 1)$ and to each other.

15 The geometric mean of $x = 2$ and $y = 8$ is $\sqrt{xy} = 4$. The arithmetic mean is larger: $\frac{1}{2}(x + y) =$ _____. This came in Example 6 from the Schwarz inequality for $v = (\sqrt{2}, \sqrt{8})$ and $w = (\sqrt{8}, \sqrt{2})$. Find $\cos\theta$ for this v and w.

16 How long is the vector $v = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector u in the same direction as v and a vector w that is perpendicular to v.

17 What are the cosines of the angles α, β, θ between the vector $(1, 0, -1)$ and the unit vectors i, j, k along the axes? Check the formula $\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1$.

Problems 18–31 lead to the main facts about lengths and angles in triangles.

18 The parallelogram with sides $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ is a rectangle. Check the Pythagoras formula $a^2 + b^2 = c^2$ which is for ***right triangles only***:

$$(\text{length of } \boldsymbol{v})^2 + (\text{length of } \boldsymbol{w})^2 = (\text{length of } \boldsymbol{v} + \boldsymbol{w})^2.$$

19 In this 90° case, $a^2 + b^2 = c^2$ also works for $\boldsymbol{v} - \boldsymbol{w}$:

$$(\text{length of } \boldsymbol{v})^2 + (\text{length of } \boldsymbol{w})^2 = (\text{length of } \boldsymbol{v} - \boldsymbol{w})^2.$$

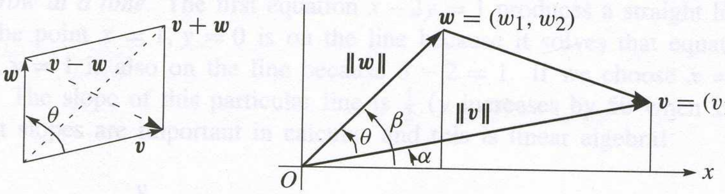
Give an example of $\boldsymbol{v}$ and $\boldsymbol{w}$ (not at right angles) for which this equation fails.

20 (Rules for dot products) These equations are simple but useful:
(1) $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ **(2)** $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ **(3)** $(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$
Use **(1)** and **(2)** with $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ to prove $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$.

21 The ***triangle inequality*** says: (length of $\boldsymbol{v} + \boldsymbol{w}$) $\leq$ (length of $\boldsymbol{v}$) + (length of $\boldsymbol{w}$). Problem 20 found $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$. Use the Schwarz inequality $\boldsymbol{v} \cdot \boldsymbol{w} \leq \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$ to turn this into the triangle inequality:

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2 \quad \text{or} \quad \|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

22 A right triangle in three dimensions still obeys $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} + \boldsymbol{w}\|^2$. Show how this leads in Problem 20 to $v_1w_1 + v_2w_2 + v_3w_3 = 0$.



23 The figure shows that $\cos\alpha = v_1/\|\boldsymbol{v}\|$ and $\sin\alpha = v_2/\|\boldsymbol{v}\|$. Similarly $\cos\beta$ is ____ and $\sin\beta$ is ____. The angle θ is $\beta - \alpha$. Substitute into the formula $\cos\beta\cos\alpha + \sin\beta\sin\alpha$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$.

24 With $\boldsymbol{v}$ and $\boldsymbol{w}$ at angle θ, the "Law of Cosines" comes from $(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w})$:

$$\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\| \, \|\boldsymbol{w}\| \cos\theta + \|\boldsymbol{w}\|^2.$$

If $\theta < 90°$ show that $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$ is larger than $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ (the third side).

25 The Schwarz inequality $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$ by algebra instead of trigonometry:

(a) Multiply out both sides of $(v_1w_1 + v_2w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Show that the difference between those sides equals $(v_1w_2 - v_2w_1)^2$. This cannot be negative since it is a square—so the inequality is true.

4 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

5 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.

6 The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line **L**). Find three solutions on **L**.

7 Move the third plane in Problem 6 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't _____ that line.

8 In Problem 6 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is _____. Find two combinations of the columns that give $\boldsymbol{b} = (2, 3, 5)$. This is only possible for $\boldsymbol{b} = (4, 6, c)$ if $c =$ _____.

9 Normally 4 "planes" in 4-dimensional space meet at a _____. Normally 4 column vectors in 4-dimensional space can combine to produce $\boldsymbol{b}$. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\boldsymbol{b} = (3, 3, 3, 2)$? What 4 equations for x, y, z, t are you solving?

Problems 10–15 are about multiplying matrices and vectors.

10 Compute each Ax by dot products of the rows with the column vector:

(a) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

11 Compute each Ax in Problem 10 as a combination of the columns:

$$10\text{(a) becomes} \quad Ax = 2 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

How many separate multiplications for Ax, when the matrix is "3 by 3"?

9 What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

10 In the xy plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y =$ ____ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c =$ ____.

Problems 11–20 study elimination on 3 by 3 systems (and possible failure).

11 Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back substitution for z, y, x.

12 Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ____ times row ____ from row ____.

13 Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

14 Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \phantom{{}-z} &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

15 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.