Solution

(1) The vectors $cv = (c, 0)$ with whole numbers $c$ are equally spaced points along the $x$ axis (the direction of $v$). They include $(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)$. Adding all vectors $dv = (0, d)$ puts a full line in the $y$ direction through those points. We have infinitely many parallel lines from $cv + dv = (\text{whole number}, \text{any number})$. These are vertical lines in the $xy$ plane, through equally spaced points on the $x$ axis.

(2) The vectors $cv$ with $c \geq 0$ fill a “half-line”. It is the positive $x$ axis, starting at $(0, 0)$ where $c = 0$. It includes $(\pi, 0)$ but not $(-\pi, 0)$. Adding all vectors $dv$ puts a full line in the $y$ direction crossing every point on that half-line. Now we have a half-plane. It is the right half of the $xy$ plane, where $x \geq 0$.

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

1. Describe geometrically (as a line, plane, …) all linear combinations of

   $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2. Draw the vectors $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $v + w$ and $v - w$ in a single $xy$ plane.

3. If $v + w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $v - w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, compute and draw $v$ and $w$.

4. From $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find the components of $3v + w$ and $v - 3w$ and $cv + dv$.

5. Compute $u + v$ and $u + v + w$ and $2u + 2v + w$ when

   $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

6. Every combination of $v = (1, -2, 1)$ and $w = (0, 1, -1)$ has components that add to _____. Find $c$ and $d$ so that $cv + dw = (4, 2, -6)$.

7. In the $xy$ plane mark all nine of these linear combinations:

   $c\begin{bmatrix} 3 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with $c = 0, 1, 2$ and $d = 0, 1, 2$. 
WORKED EXAMPLES

1.2 A For the vectors \( \mathbf{v} = (3, 4) \) and \( \mathbf{w} = (4, 3) \) test the Schwarz inequality on \( \mathbf{v} \cdot \mathbf{w} \) and the triangle inequality on \( \| \mathbf{v} + \mathbf{w} \| \). Find \( \cos \theta \) for the angle between \( \mathbf{v} \) and \( \mathbf{w} \). When will we have equality \( |\mathbf{v} \cdot \mathbf{w}| = \|\mathbf{v}\| \|\mathbf{w}\| \) and \( \|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\| \)?

Solution The dot product is \( \mathbf{v} \cdot \mathbf{w} = (3)(4) + (4)(3) = 24 \). The length of \( \mathbf{v} \) is \( \|\mathbf{v}\| = \sqrt{9 + 16} = 5 \) and also \( \|\mathbf{w}\| = 5 \). The sum \( \mathbf{v} + \mathbf{w} = (7, 7) \) has length \( \|\mathbf{v} + \mathbf{w}\| = 7\sqrt{2} \approx 9.9 \).

**Schwarz inequality** \( |\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\| \) is \( 24 < 25 \).

**Triangle inequality** \( \|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\| \) is \( 7\sqrt{2} < 10 \).

**Cosine of angle** \( \cos \theta = \frac{24}{25} \) (Thin angle!)

If one vector is a multiple of the other as in \( \mathbf{w} = -2\mathbf{v} \), then the angle is 0° or 180° and \( |\cos \theta| = 1 \) and \( |\mathbf{v} \cdot \mathbf{w}| \) equals \( \|\mathbf{v}\| \|\mathbf{w}\| \). If the angle is 0°, as in \( \mathbf{w} = 2\mathbf{v} \), then \( \|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\| \). The triangle is flat.

1.2 B Find a unit vector \( \mathbf{u} \) in the direction of \( \mathbf{v} = (3, 4) \). Find a unit vector \( \mathbf{U} \) perpendicular to \( \mathbf{u} \). How many possibilities for \( \mathbf{U} \)?

Solution For a unit vector \( \mathbf{u} \), divide \( \mathbf{v} \) by its length \( \|\mathbf{v}\| = 5 \). For a perpendicular vector \( \mathbf{V} \) we can choose \((-4, 3)\) since the dot product \( \mathbf{v} \cdot \mathbf{V} \) is \((3)(-4) + (4)(3) = 0\).

For a unit vector \( \mathbf{U} \), divide \( \mathbf{V} \) by its length \( \|\mathbf{V}\| \):

\[
\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(3, 4)}{5} = \left( \frac{3}{5}, \frac{4}{5} \right) \quad \mathbf{U} = \frac{\mathbf{V}}{\|\mathbf{V}\|} = \frac{(-4, 3)}{5} = \left( \frac{-4}{5}, \frac{3}{5} \right)
\]

The only other perpendicular unit vector would be \( -\mathbf{U} = \left( \frac{4}{5}, -\frac{3}{5} \right) \).

**Problem Set 1.2**

1. Calculate the dot products \( \mathbf{u} \cdot \mathbf{v} \) and \( \mathbf{u} \cdot \mathbf{w} \) and \( \mathbf{v} \cdot \mathbf{w} \) and \( \mathbf{w} \cdot \mathbf{v} \):

\[
\mathbf{u} = \begin{bmatrix} -\frac{6}{8} \\ \frac{3}{4} \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}
\]

2. Compute the lengths \( \|\mathbf{u}\| \) and \( \|\mathbf{v}\| \) and \( \|\mathbf{w}\| \) of those vectors. Check the Schwarz inequalities \( |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \) and \( |\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\| \).

3. Find unit vectors in the directions of \( \mathbf{v} \) and \( \mathbf{w} \) in Problem 1, and the cosine of the angle \( \theta \). Choose vectors that make 0°, 90°, and 180° angles with \( \mathbf{w} \).

4. Find unit vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) in the directions of \( \mathbf{v} = (3, 1) \) and \( \mathbf{w} = (2, 1, 2) \). Find unit vectors \( \mathbf{U}_1 \) and \( \mathbf{U}_2 \) that are perpendicular to \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \).
For any unit vectors \( \mathbf{v} \) and \( \mathbf{w} \), find the dot products (actual numbers) of
(a) \( \mathbf{v} \) and \( -\mathbf{v} \)  
(b) \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \)  
(c) \( \mathbf{v} - 2\mathbf{w} \) and \( \mathbf{v} + 2\mathbf{w} \)

Find the angle \( \theta \) (from its cosine) between
(a) \( \mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)  
(b) \( \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \)  
(c) \( \mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \)  
(d) \( \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \) and \( \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \)

(a) Describe every vector \( \mathbf{w} = (w_1, w_2) \) that is perpendicular to \( \mathbf{v} = (2, -1) \).  
(b) The vectors that are perpendicular to \( V = (1, 1, 1) \) lie on a  
(c) The vectors that are perpendicular to \( (1, 1, 1) \) and \( (1, 2, 3) \) lie on a

True or false (give a reason if true or a counterexample if false):
(a) If \( \mathbf{u} \) is perpendicular (in three dimensions) to \( \mathbf{v} \) and \( \mathbf{w} \), then \( \mathbf{v} \) and \( \mathbf{w} \) are parallel.  
(b) If \( \mathbf{u} \) is perpendicular to \( \mathbf{v} \) and \( \mathbf{w} \), then \( \mathbf{u} \) is perpendicular to \( \mathbf{v} + 2\mathbf{w} \).  
(c) If \( \mathbf{u} \) and \( \mathbf{v} \) are perpendicular unit vectors then \( \| \mathbf{u} - \mathbf{v} \| = \sqrt{2} \)

The slopes of the arrows from \((0, 0)\) to \((v_1, v_2)\) and \((w_1, w_2)\) are \(v_2/v_1\) and \(w_2/w_1\). If the product \(v_2w_2/v_1w_1\) of those slopes is \(-1\), show that \(\mathbf{v} \cdot \mathbf{w} = 0\) and the vectors are perpendicular.

Draw arrows from \((0, 0)\) to the points \( \mathbf{v} = (1, 2) \) and \( \mathbf{w} = (-2, 1) \). Multiply their slopes. That answer is a signal that \( \mathbf{v} \cdot \mathbf{w} = 0 \) and the arrows are

If \( \mathbf{v} \cdot \mathbf{w} \) is negative, what does this say about the angle between \( \mathbf{v} \) and \( \mathbf{w} \)? Draw a 2-dimensional vector \( \mathbf{v} \) (an arrow), and show where to find all \( \mathbf{w} \)'s with \( \mathbf{v} \cdot \mathbf{w} < 0 \).

With \( \mathbf{v} = (1, 1) \) and \( \mathbf{w} = (1, 5) \) choose a number \( c \) so that \( \mathbf{w} - c\mathbf{v} \) is perpendicular to \( \mathbf{v} \). Then find the formula that gives this number \( c \) for any nonzero \( \mathbf{v} \) and \( \mathbf{w} \).

Find two vectors \( \mathbf{v} \) and \( \mathbf{w} \) that are perpendicular to \((1, 0, 1)\) and to each other.

Find three vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) that are perpendicular to \((1, 1, 1, 1)\) and to each other.

The geometric mean of \( x = 2 \) and \( y = 8 \) is \( \sqrt{xy} = 4 \). The arithmetic mean is larger: \( \frac{1}{2}(x+y) = 5 \). This came in Example 6 from the Schwarz inequality for \( \mathbf{v} = (\sqrt{2}, \sqrt{8}) \) and \( \mathbf{w} = (\sqrt{8}, \sqrt{2}) \). Find \( \cos \theta \) for this \( \mathbf{v} \) and \( \mathbf{w} \).

How long is the vector \( \mathbf{v} = (1, 1, \ldots, 1) \) in \( n \) dimensions? Find a unit vector \( \mathbf{u} \) in the same direction as \( \mathbf{v} \) and a vector \( \mathbf{w} \) that is perpendicular to \( \mathbf{v} \).

What are the cosines of the angles \( \alpha, \beta, \theta \) between the vector \((1, 0, -1)\) and the unit vectors \( i, j, k \) along the axes? Check the formula \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \theta = 1 \).
Problems 18–31 lead to the main facts about lengths and angles in triangles.

18 The parallelogram with sides \( v = (4, 2) \) and \( w = (-1, 2) \) is a rectangle. Check the Pythagoras formula \( a^2 + b^2 = c^2 \) which is for right triangles only:

\[
\text{(length of } v \text{)}^2 + \text{(length of } w \text{)}^2 = \text{(length of } v + w \text{)}^2.
\]

19 In this \( 90^\circ \) case, \( a^2 + b^2 = c^2 \) also works for \( v - w \):

\[
\text{(length of } v \text{)}^2 + \text{(length of } w \text{)}^2 = \text{(length of } v - w \text{)}^2.
\]

Give an example of \( v \) and \( w \) (not at right angles) for which this equation fails.

20 (Rules for dot products) These equations are simple but useful:

1. \( v \cdot w = w \cdot v \)
2. \( u \cdot (v + w) = u \cdot v + u \cdot w \)
3. \( (cv) \cdot w = c(v \cdot w) \)

Use (1) and (2) with \( u = v + w \) to prove \( \|v + w\|^2 = v \cdot v + 2v \cdot w + w \cdot w \).

21 The triangle inequality says: \( \|v + w\| \leq \|v\| + \|w\| \). Problem 20 found \( \|v + w\|^2 = \|v\|^2 + 2v \cdot w + \|w\|^2 \). Use the Schwarz inequality \( v \cdot w \leq \|v\| \|w\| \) to turn this into the triangle inequality:

\[
\|v + w\|^2 \leq (\|v\| + \|w\|)^2 \quad \text{or} \quad \|v + w\| \leq \|v\| + \|w\|.
\]

22 A right triangle in three dimensions still obeys \( \|v\|^2 + \|w\|^2 = \|v + w\|^2 \). Show how this leads in Problem 20 to \( v_1w_1 + v_2w_2 + v_3w_3 = 0 \).

23 The figure shows that \( \cos \alpha = v_1 / \|v\| \) and \( \sin \alpha = v_2 / \|v\| \). Similarly \( \cos \beta \) is \( \frac{w_1}{\|w\|} \) and \( \sin \beta \) is \( \frac{w_2}{\|w\|} \). The angle \( \theta \) is \( \beta - \alpha \). Substitute into the formula \( \cos \beta \cos \alpha + \sin \beta \sin \alpha \) for \( \cos(\beta - \alpha) \) to find \( \cos \theta = v \cdot w / \|v\| \|w\| \).

24 With \( v \) and \( w \) at angle \( \theta \), the “Law of Cosines” comes from \( (v - w) \cdot (v - w) \):

\[
\|v - w\|^2 = \|v\|^2 - 2\|v\| \|w\| \cos \theta + \|w\|^2.
\]

If \( \theta < 90^\circ \) show that \( \|v\|^2 + \|w\|^2 \) is larger than \( \|v - w\|^2 \) (the third side).

25 The Schwarz inequality \( |v \cdot w| \leq \|v\| \|w\| \) by algebra instead of trigonometry:

(a) Multiply out both sides of \( (v_1w_1 + v_2w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2) \).

(b) Show that the difference between those sides equals \( (v_1w_2 - v_2w_1)^2 \). This cannot be negative since it is a square—so the inequality is true.
4. If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

5. Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.

6. The first of these equations plus the second equals the third:

$$x + y + z = 2$$
$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5.$$  

The first two planes meet along a line. The third plane contains that line, because if $x, y, z$ satisfy the first two equations then they also _________. The equations have infinitely many solutions (the whole line $L$). Find three solutions on $L$.

7. Move the third plane in Problem 6 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—why not? The first two planes meet along the line $L$, but the third plane doesn’t ________ that line.

8. In Problem 6 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is ________. Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c = ______$. 

9. Normally 4 “planes” in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-dimensional space can combine to produce $b$. What combination of $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? What 4 equations for $x, y, z, t$ are you solving?

Problems 10–15 are about multiplying matrices and vectors.

10. Compute each $Ax$ by dot products of the rows with the column vector:

   (a) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

11. Compute each $Ax$ in Problem 10 as a combination of the columns:

   10(a) becomes $Ax = 2 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$.

   How many separate multiplications for $Ax$, when the matrix is “3 by 3”? 


9 What test on $b_1$ and $b_2$ decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

\[ 3x - 2y = b_1 \]
\[ 6x - 4y = b_2. \]

10 In the $xy$ plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y = \_\_\_\_$ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c = \_\_\_\_\_\_$.

Problems 11–20 study elimination on 3 by 3 systems (and possible failure).

11 Reduce this system to upper triangular form by two row operations:

\[
\begin{align*}
2x + 3y + z &= 8 \\
4x + 7y + 5z &= 20 \\
-2y + 2z &= 0.
\end{align*}
\]

Circle the pivots. Solve by back substitution for $z, y, x$.

12 Apply elimination (circle the pivots) and back substitution to solve

\[
\begin{align*}
2x - 3y &= 3 \\
4x - 5y + z &= 7 \\
2x - y - 3z &= 5.
\end{align*}
\]

List the three row operations: Subtract ____ times row ____ from row ____.

13 Which number $d$ forces a row exchange, and what is the triangular system (not singular) for that $d$? Which $d$ makes this system singular (no third pivot)?

\[
\begin{align*}
2x + 5y + z &= 0 \\
4x + dy + z &= 2 \\
y - z &= 3.
\end{align*}
\]

14 Which number $b$ leads later to a row exchange? Which $b$ leads to a missing pivot? In that singular case find a nonzero solution $x, y, z$.

\[
\begin{align*}
x + by &= 0 \\
x - 2y - z &= 0 \\
y + z &= 0.
\end{align*}
\]

15 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.