Lecture 04:
Systems of Linear Equations #3:

Outline:

1) Matrix-Matrix Operations (the rules of the game)
   - Matrix Multiplication: AB
   - Block Matrices
   - Proof of Associative law A(B<u>c</u>)=(AB)<u>c</u>
   - Computational Costs
   - Matrix Powers: A^p
2) The Matrix Inverse: A^{-1} (square matrices)
   - Fun Facts about A^{-1}
   - The inverse of simple matrices (I,D,E,P)
   - The inverse of General matrices: Gauss-Jordan Elimination

General rules of Matrix-Matrix Operations

Matrix Matrix Multiplication: C=AB

General rules of Matrix-Matrix Operations

Matrix Matrix Multiplication: C=AB

Examples:

General rules of Matrix-Matrix Operations

Properties of Matrix-Matrix Multiplication:

Block Matrices:

Examples:

Proof of Associative property for Matrix Multiplication

Theorem: Matrix Mult is associative.
If A,B,C are matrices of appropriate shapes, then A(BC)=(AB)C

Proof (sketch): Show $A(B\underline{c})=(AB)\underline{c}$

Another important Digression: Operation costs

Matrix-vector and Matrix-Matrix multiplication (order matters!)

Matlab Demo:

General rules of Matrix-Matrix Operations

Matrix Powers: A^p

The Matrix Inverse: A^{-1}

Definition: A square matrix A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = I \text{ and } AA^{-1}=I$$

Note: Not all square matrices are invertible!

The Matrix Inverse: A^{-1}

6 Fun Facts about A^{-1}:

1) **Given A nxn: A^{-1} exists if and only if elimination produces *n* pivots. If there are < *n* pivots, A is singular. (to be proved)**

2) **If A^{-1} exists, it is *unique***

3) **If A^{-1} exists, solutions to Ax=b are unique.**

The Matrix Inverse: A^{-1}

6 Fun Facts about A^{-1} cont'd.:

4) **if there exists a vector $\underline{x}\neq\underline{0}$ such that $A\underline{x}=\underline{0}$, then A is singular.**
($A\underline{x}=\underline{0}$ for non-zero $\underline{x}$ implies that there is a linear combination of the columns of A that add up to zero...)

5) **The inverse of a 2x2 matrix (if it exists) is**

6) **Product Rule for Inverses! if A and B are both invertible, then so is their product AB with inverse**

Inverses of simple Matrices:

The identity matrix:

Diagonal Matrices:

Elementary Elimination Matrices E_{ij} :

Even Permutation Matrices:

Inverses of General Square Matrices:
Gauss-Jordan Elimination

The Big Picture

Inverses of General Square Matrices:
Gauss-Jordan Elimination

Step 1: Form the block matrix [A I]

Step 2: Eliminate Down (Gauss)

Step 3: Eliminate Up (Jordan)

Step 4: Divide by the Pivots

Inverses of General Square Matrices:
Gauss-Jordan Elimination

A 3x3 Example (ugh):

Big Point!:

To solve $A\underline{x}=\underline{b}$ use Gaussian Elimination, not $\underline{x}=A^{-1}\underline{b}$

Being Invertible is important...

finding A^{-1} is less important

Gauss Elimination is a sufficient test for invertibility!

Theorem: A^{-1} exists if and only if A has a full set of n pivots.

Proof:

1) Given n pivots show you can find A^{-1}

2) if AC=I then A must have n pivots