

Lecture 05:

Systems of Linear Equations #4: A^{-1} and the LU decomposition

Outline:

- 1) The Matrix Inverse: A^{-1} (cont'd)
 - The Mechanics: Gauss-Jordan Elim (3x3 example)
 - The Theory: Proof: invertibility iff n pivots
- 2) The LU factorization: $A=LU$ (the right way to solve $A\mathbf{x}=\mathbf{b}$)
 - The idea of factorizations
 - Finding $L=E^{-1}$
 - Examples of $A=LU$
 - Understanding the LU factorization
 - Using $A=LU$ to solve $A\mathbf{x}=\mathbf{b}$
 - Operation Costs
 - row swaps and $PA=LU$
 - Matlab Examples

**Inverses of General Square Matrices:
Gauss-Jordan Elimination**

The Big Picture

Step 1: Form the block matrix $[A \ I]$

Step 2: Eliminate Down (Gauss)

Step 3: Eliminate Up (Jordan)

Step 4: Divide by the Pivots

**Inverses of General Square Matrices:
Gauss-Jordan Elimination**

A 3x3 Example (ugh): $A = [\ 1 \ 4 \ 3 \ ; \ -1 \ -2 \ 0; \ 2 \ 2 \ 3]$

Big Point!:

To solve $A\mathbf{x}=\mathbf{b}$ use **Gaussian Elimination**, **not** $\mathbf{x}=A^{-1}\mathbf{b}$

Being Invertible is important...

finding A^{-1} is less important

But Elimination is both the mechanism and the proof of invertibility

Gauss Elimination is a sufficient test for invertibility!

Theorem: A^{-1} exists **if and only if** A has a full set of n pivots.

Proof:

- 1) if A has n pivots then A^{-1} can be found using Elimination

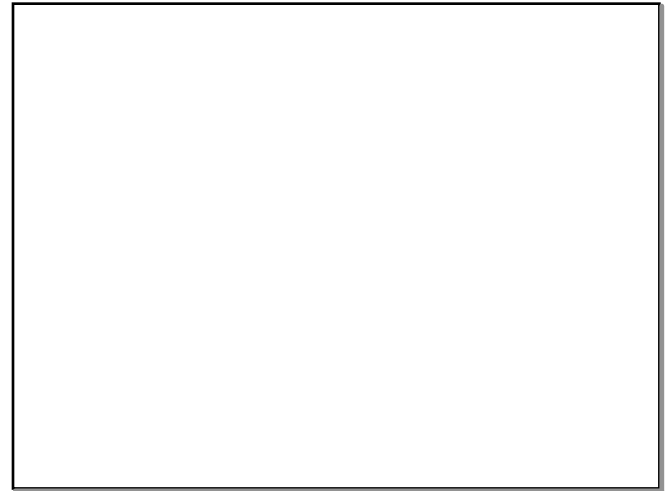
Gauss Elimination is a sufficient test for invertibility!

Theorem: A^{-1} exists if and only if A has a full set of n pivots.

Proof:

2) if A is invertible then A must have n pivots (proof by contradiction)

Assume: A is invertible **and** A has $< n$ pivots



The LU Factorization:

The General Idea of factorization:

Take a general matrix A and **factor** it into the product of two (or more) better behaved matrices.

For solving $Ax=b$: **triangular** matrices are good so the LU factorization factors A into

$$A=LU$$

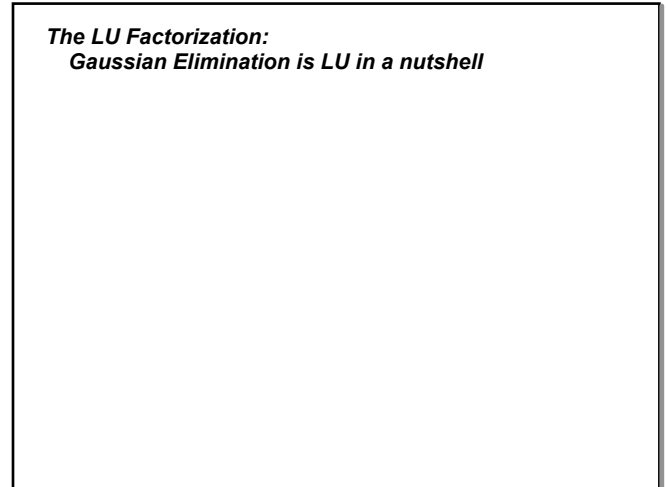
where

L is lower triangular (1's on the diagonal, l_{ij} in their place)
 U is upper triangular (our old friend U)

Question: Where does L come from?

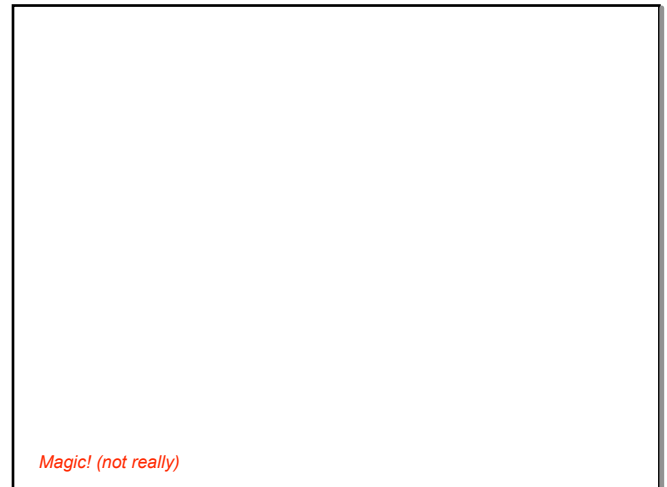
The LU Factorization:

Gaussian Elimination is LU in a nutshell



The LU Factorization:

Show that $E^{-1} = L$ (lower triangular with 1 on the diag)



Magic! (not really)

The LU Factorization:
 Example: $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

The LU Factorization:
 Why it works:
 GE (and the LU) simply decompose the rows of A into a linear combination of the rows of U .

Using the LU to solve $Ax=b$

Points:

- 1) Splits $Ax=b$ into **two** triangular problems ($Lc=b$, $Ux=c$)
- 2) Easily solved by Forward then back substitution
- 3) No more expensive than standard Gaussian Elimination
- 4) Can solve multiple RHS's without refinding U
- 5) Is the standard algorithm for numerical direct solvers (e.g. Matlab's $x=A\b{b}$)

Using the LU to solve $Ax=b$

Example: $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$, $x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Using the LU to solve $Ax=b$

Operation Costs:

Factorization:

Solve:

Comparison to GJE for A^{-1} :

The LU factorization: Final comments

Operation Costs:
 For special matrices can be significantly better than $n^3/3$
 Always better than $x=A^{-1}b$

MATLAB dos and don'ts:
DON'T use: $x=inv(A)*b$
DO use: $x=A\b{b}$

Row Exchanges (Partial Pivoting):

- 1) Row exchanges make things a bit more complicated
- 2) For numerical stability, Matlab ALWAYS exchanges rows
- 3) Book keeping is more complicated but the LU can always be generalized to

$PA=LU$

where P is a permutation matrix

Comparison to GJE for A^{-1} :

The LU factorization with Row Exchanges (PA=LU)

Example: [1 0 1; 0 1 2; 0 1 1]

Comparison to GJE for A^{-1} :

