Lecture 05:

Systems of Linear Equations #4: A^{-1} and the LU decomposition

Outline:

1) The Matrix Inverse: A^{-1} (cont'd)
   - The Mechanics: Gauss-Jordan Elim (3x3 example)
   - The Theory: Proof: invertibility iff n pivots
2) The LU factorization: A=LU (the right way to solve $A\underline{x}=\underline{b}$)
   - The idea of factorizations
   - Finding $L=E^{-1}$
   - Examples of A=LU
   - *Understanding* the LU factorization
   - Using A=LU to solve $A\underline{x}=\underline{b}$
   - Operation Costs
   - row swaps and PA=LU
   - Matlab Examples

Inverses of General Square Matrices:

Gauss-Jordan Elimination

The Big Picture

Step 1: Form the block matrix [A I]

Step 2: Eliminate Down (Gauss)

Step 3: Eliminate Up (Jordan)

Step 4: Divide by the Pivots

Inverses of General Square Matrices:

Gauss-Jordan Elimination

A 3x3 Example (ugh): A = [1 4 3 ; -1 -2 0; 2 2 3]

Big Point!:

To solve $A\underline{x}=\underline{b}$ use Gaussian Elimination, not $\underline{x}=A^{-1}\underline{b}$

Being Invertible is important...

finding A^{-1} is less important

But Elimination is both the mechanism and the proof of invertibility

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Gauss Elimination is a sufficient test for invertibility!

Theorem: A^{-1} exists if and only if A has a full set of n pivots.

Proof:

1) if A has n pivots then A^{-1} can be found using Elimination

Gauss Elimination is a sufficient test for invertibility!

Theorem: A^{-1} exists if and only if A has a full set of n pivots.

Proof:

2) if A is invertible then A must have n pivots (proof by contradiction)

Assume: A is invertible and A has < n pivots

The LU Factorization:

The General Idea of factorization:

Take a general matrix A and factor it into the product of two (or more) better behaved matrices.

For solving Ax=b: triangular matrices are good so the LU factorization factors A into

$$A=LU$$

where

L is lower triangular (1's on the diagonal, l_{ij} in their place)
U is upper triangular (our old friend U)

Question: Where does L come from?

The LU Factorization:
Gaussian Elimination is LU in a nutshell

The LU Factorization:
Show that $E^{-1} = L$ (lower triangular with 1 on the diag)

Magic! (not really)

The LU Factorization:

Example: A = [1 4 3 ; -1 -2 0; 2 2 3]

The LU Factorization:

Why it works:

GE (and the LU) simply decompose the rows of A into a linear combination of the rows of U.

Using the LU to solve A$\underline{x}$=$\underline{b}$

Points:

- ***1) Splits Ax=b into two triangular problems (L$\underline{c}$=$\underline{b}$, U$\underline{x}$=$\underline{c}$)***
- ***2) Easily solved by Forward then back substitution***
- ***3) No more expensive than standard Gaussian Elimination***
- ***4) Can solve multiple RHS's without refinding U***
- ***5) Is the standard algorithm for numerical direct solvers***
  - ***(e.g. Matlab's x=A\b)***

Using the LU to solve A$\underline{x}$=$\underline{b}$

Example: A=[1 1 0; 4 6 1; -2 2 0], $\underline{x}$=[-1 1 0]'

Using the LU to solve A$\underline{x}$=$\underline{b}$

Operation Costs:

Factorization:

Solve:

Comparison to GJE for A^{-1}:

The LU factorization: Final comments

Operation Costs:

For special matrices can be significantly better than $n^3/3$

Always better than $\underline{x}=A^{-1}\underline{b}$

MATLAB dos and don'ts:

DON'T use: $\underline{x}$=inv(A)*$\underline{b}$
DO use: $\underline{x}$=A\\$\underline{b}$

Row Exchanges (Partial Pivoting):

- ***1) Row exchanges make things a bit more complicated***
- ***2) For numerical stability, Matlab ALWAYS exchanges rows***
- ***3) Book keeping is more complicated but the LU can always be generalized to***

PA=LU

where P is a permutation matrix

Comparison to GJE for A^{-1}:

The LU factorization with Row Exchanges (PA=LU)

Example: [1 0 1; 0 1 2; 0 1 1]

Comparison to GJE for A^{-1}: