Lecture 06:
Systems of Linear Equations #5: Everything else

Outline:

1) The LU factorization: final comments
   - Row swaps: PA=LU
     - Permutation Matrices and an Example
   - A "real" problem: SpringDemo and Matlab
     - Sparse, TriDiagonal matrices
     - LU vs inv(A);

2) One last Operation: The matrix transpose A^T
   - symmetric matrices $A^T=A$
   - product Rule $(AB)^T=B^TA^T$
   - symmetry of R^TR and RR^T

LU Factorization and row exchanges: PA=LU

The Problem:
A full description of Gaussian Elimination includes Permutation matrices

The Fix:
In General you can't know the order of permutations before you begin, but you can track permutations as you proceed such that at the end, you can permute A once such that PA=LU.

Two Matlab approaches (version 7+)

[L,U,P]=lu(A); % such that P*A=L*U

-or-

[L,U,p]=lu(A,'vector') % such that A(p,:)=L*U

LU Factorization and row exchanges: PA=LU

For small problems, we'll just permute first then find the LU (and let Matlab handle the hard stuff)

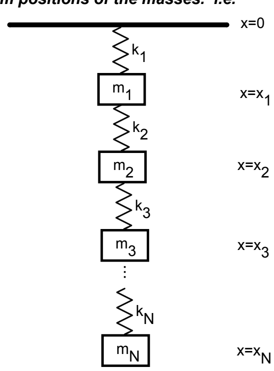
Example: A = [0 1 2 ; 1 0 1; 0 1 1] (several choices for P)

LU Factorization and row exchanges: PA=LU

Solving A*x*=*b* using PA=LU

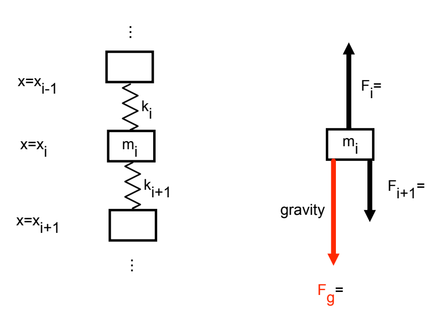
The LU in action: the SpringBlock Demo using Matlab

The Problem: given N masses (each of mass m_i)connected by N springs (with spring constants k_i) under gravity, find the equilibrium positions of the masses. i.e.



The LU in action: the SpringBlock Demo using Matlab

Force Balance on each block, leads to a system of linear equations

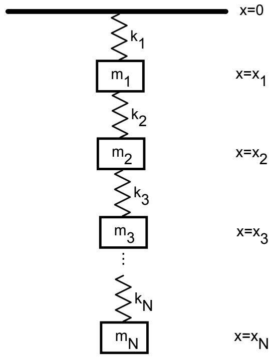


Equilibrium implies:

The LU in action: the SpringBlock Demo using Matlab

Force Balance on each block, leads to a system of linear equations

$-k_i x_{i-1} + (k_i + k_{i+1}) x_i - k_{i+1} x_{i+1} = m_i g$



The LU in action: the SpringBlock Demo using Matlab

Systems of linear equations can always be written $A\underline{x}=\underline{b}$ (or $K\underline{x}=\underline{m}$)

$-k_i x_{i-1} + (k_i + k_{i+1}) x_i - k_{i+1} x_{i+1} = m_i g$

Note: coupled 1-D system leads to a Sparse, Tridiagonal, Symmetric matrix A

The LU in action: the SpringBlock Demo using Matlab

Special Case, $k_i = m_i = 1$ (all masses and springs are equal)

$-k_i x_{i-1} + (k_i + k_{i+1}) x_i - k_{i+1} x_{i+1} = m_i g$

The LU in action: the SpringBlock Demo using Matlab

Special Case, $k_i = m_i = 1$ (all masses and springs are equal)

Solve using Matlab:

```
N=5;
k=ones(N,1);
m=ones(N,1);
[x,A]=springDemo(k,m)
seeLUvsInv(A);  % compare LU(A) to inv(A)
```

Themes and variations:

```
N=100;  % large numbers
k=rand(N,1); % random spring values
             %(can lead to near singular matrices)
m=zeros(N,1); m(i)=1;  % single point mass
```

The LU in action: the SpringBlock Demo using Matlab

Big Points:

Coupled Dynamics often lead to large sparse linear systems

LU of a tri-diagonal system is still tri-diagonal (fill in)

A^{-1} for a tri-diagonal system can be dense (why?)

The LU in action: the SpringBlock Demo using Matlab

Big Points:

Operation costs for tri-diagonal A

- *x=A\b (LU decomposition) O(N)*
- *x=inv(A)*b (inverse) $O(N^3)$ for the inverse and $O(N^2)$ for $A^{-1}b$*

Multi-dimensional problems (2-D, 3-D) still sparse but much more expensive by "direct methods"...but there are many other ways to solve $A\underline{x}=\underline{b}$ (but not in this class)

Final Detail before the big picture:

The Matrix Transpose A^T

Vector transpose x^T:
transforms a column vector into a row vector (and vice versa)

Vectors are just skinny matrices (m x 1), therefore...

The Matrix Transpose A^T transforms columns of A into ...?

Examples:

Symmetric Matrices:

Definition: A square Matrix A is symmetric if $A^T=A$ (i.e. the rows are the same as the columns)

Examples:

It's not obvious yet but symmetric matrices have important and special properties.

Rules of the Matrix Transpose:

Addition:

Repeated Transpose:

Product Rule: (very important!)

Proof: start by showing $(Ax)^T=x^TA^T$...

Onwards to the big ideas:

Vector Spaces and subspaces:
Linear Independence
Bases and Dimension:
The deeper meaning of $A\underline{x}=\underline{b}$:
the 4 fundamental subspaces of a matrix A

More Fun Facts using the matrix transpose:

Inverse of A^T: prove that $(A^T)^{-1}=(A^{-1})^T$

Show A^TA and AA^T are square symmetric matrices (but not equal)

Symmetric Matrices can be factored as $A=LDL^T$ (where D is a diagonal matrix of pivots)