Lecture 07:
Vector Spaces and Subspaces #1

Outline:

1) Intro: toward a deeper understanding of $A\underline{x}=\underline{b}$

2) Vector Spaces
   Definition and Rules
   Real Vector Spaces R^n
   Other Vector Spaces (C^n, $R^{m \times n}$, Z, F)
3) Vector Subspaces
   Definition
   Lots of Examples
4) Fundamental Subspaces associated with a matrix A
   The Column Space C(A)
   The Null Space N(A)

Linear Vector Spaces: an abstract definition

Formal Definition:
A vector space is a set of objects (vectors) and rules of vector addition and scalar multiplication that satisfy the following axioms

1) $\underline{x} + \underline{y} = \underline{y} + \underline{x}$
2) $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x}+\underline{y}) + \underline{z}$
3) There exists a unique zero vector such that $\underline{x}+\underline{0}=\underline{x}$
4) For every vector $\underline{x}$ there is a unique vector $-\underline{x}$ s.t. $\underline{x} + (-\underline{x}) = \underline{0}$
5) 1 times $\underline{x}$ equals $\underline{x}$
6) $(c_1c_2)\underline{x}=c_1(c_2\underline{x})$
7) $c(\underline{x} + \underline{y}) = c\underline{x} + c\underline{y}$
8) $(c_1 + c_2)\underline{x} = c_1\underline{x} + c_2\underline{x}$

The important consequence of these axioms is that the set is closed under vector addition and scalar multiplication (i.e. any linear combination of vectors chosen from the space remains a member of the space)

Linear Vector Spaces: a concrete example
the set of real vectors R^n

Definition:
The space R^n is the set of all column vectors $\underline{v}$ with n real components (together with the standard rules of vector addition and scalar multiplication)

Examples:

Point: Vector spaces are really "spaces" (geometric objects)

Linear Vector Spaces: Other Examples

Complex vectors:

mxn real matrices:

The Zero Vector Space:

Linear Vector Spaces: Other Examples (not just vectors!)

Real Functions f(x):

The space of Piecewise Linear Functions (connect the dots...):

Vector Subspaces:

Definition: A subspace S of a vector space is a set of vectors (including $\underline{0}$) that satisfies two requirements: if $\underline{v},\underline{w} \in S$ then

1) $\underline{v} + \underline{w} \in S$
2) $c\underline{v} \in S$

(i.e. the subspace is also closed to addition and scalar multiplication -or- all linear combinations of vectors in the subspace remain in the subspace)

Vector Subspaces: Examples Galore!
(subspace or not?):

Example 1): The set of all vectors in R^3 whose last component is zero

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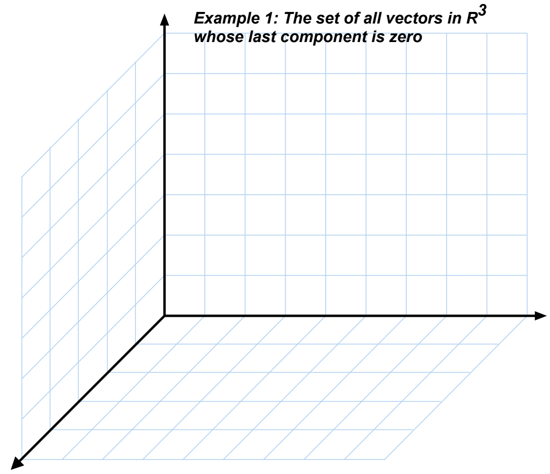

Vector Subspaces: Examples Galore!

Example 2): The set of all vectors in R^2 whose components sum to 0

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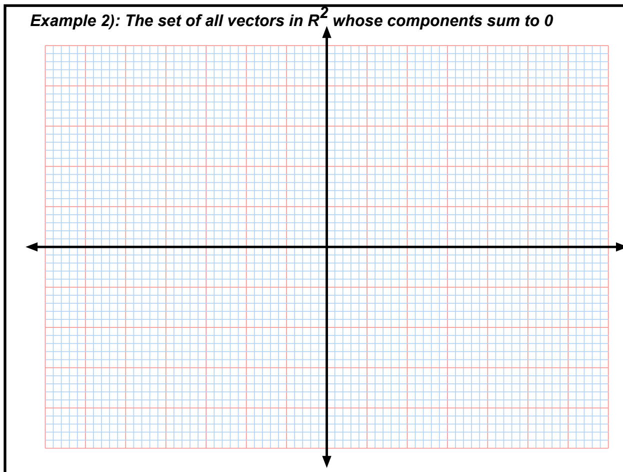
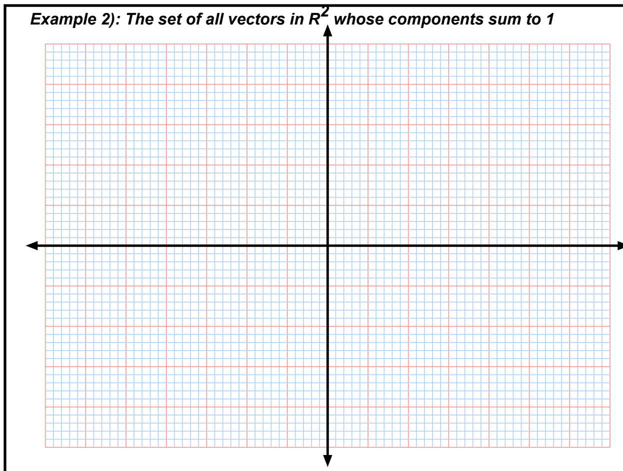


Vector Subspaces: Examples Galore!
(am I a subspace or not):

Example 3): The set of all vectors in R^2 whose components sum to 1

Example 2): The set of all vectors in R^2 whose components sum to 1



Vector Subspaces: Subspace or not...
Some basic rules for testing subspaces

1) To prove a set of vectors is a subspace you must show it is closed for all vectors in the set.
2) To prove a set of vectors is not a subspace you only need a single counter example.

3) All vector spaces and sub-spaces must include the zero vector! so check for this first.

A Few last examples: Subspace or not?

A) the set of all vectors in R^2 with components ≥ 0

B) the set of all symmetric 2x2 matrices

C) the set of all invertible 2x2 matrices

All permissible subspaces in R^3

1) Linear combinations of 1 vector
Geometry?

2) Linear combinations of 2 vectors
Geometry?

3)?

4)?

The 4 Fundamental Subspaces of a matrix A
Control the existence and uniqueness of solutions to $A\underline{x}=\underline{b}$

The Column Space:

The Null Space:

The Row Space:

The Left Null Space:

The Column Space:

Definition:
The Column Space of a Matrix A is the vector subspace formed by all linear combinations of the columns of A.

Examples:

The Column Space:

Properties: for $A \in R^{m \times n}$

$C(A) \subset R^{?}$

$C(A) \equiv A\underline{x} \;\; \forall \; \underline{x} \in R^{?}$

C(A) controls the ___________________ of solutions to $A\underline{x}=\underline{b}$

i.e. $A\underline{x}=\underline{b}$ has a solution iff ___________________

The Null Space: N(A)

Definition:
The Null Space of a Matrix A is the vector subspace formed by all solutions $\underline{x}$ to $A\underline{x}=\underline{0}$ (i.e. is all linear combination of columns of A that cancel to zero)

Proof that N(A) is a subspace:

The Null Space: N(A)

Simple Examples:

The importance of the Null Space: uniqueness of solutions to A$\underline{x}$=$\underline{b}$

The Null Space:

Properties: for $A \in R^{m \times n}$

$N(A) \subset R^?$

N(A) ≡ Z if A is ______________________

if N(A) ≠ Z then Ax=b has _____________solutions

i.e. N(A) controls the ________________ of solutions to A$\underline{x}$=$\underline{b}$

The Null Space: Algorithms for finding N(A)

1) Gaussian Elimination on mxn matrices

The Null Space: Algorithms for finding N(A)

1) Gaussian-Jordan Elimination to reduced row Echelon Form (rref(A))