Lecture 09:
The Big picture

Outline:

1) The general solution to A$\underline{x}$=$\underline{b}$
2) The ***BIG*** ideas
   A) Linear Independence
   B) Span
   C) *Basis*
   D) Dimension
3) Bases and Dimensions for the 4 Fundamental Subspaces of a mxn Matrix A
   C(A), N(A), $C(A^T)$, $N(A^T)$
4) Orthogonality of the 4 subspaces

The General solution to A$\underline{x}$=$\underline{b}$

Given A in R^{mxn} and b in R^m, we want to find *all* solutions to Ax=b (if they exist)

General Algorithm:

1) use Gauss-Jordan Elimination to reduce [A $\underline{b}$] to [R $\underline{d}$]

2) determine if the rhs is *consistent* (i.e. $\underline{d}$ in C(R) or $\underline{b}$ in C(A))

   A) if not consistent, there is no solution
   B) if consistent then find one particular solution such that
      $R\underline{x}_p$=d (or $A\underline{x}_p$=$\underline{b}$)
      (combination of pivot columns of R that add to form $\underline{d}$)

3) Find all special solutions to A$\underline{x}$=$\underline{0}$ and form null space matrix N

4) the general solution is $\underline{x}=\underline{x}_p+\underline{x}_N$ where $\underline{x}_N=N\underline{c}$ is all vectors in the Null space N(A)

The General solution to A$\underline{x}$=$\underline{b}$

Example #1: A=[1 2 1 0 1 ; 2 4 1 0 0; 1 2 0 1 -4], $\underline{b}$=[1 1 1]'

The General solution to A$\underline{x}$=$\underline{b}$

Example #2: A=[1 1 ; 1 -2 ; -2 1], $\underline{b}$=[b_1 b_2 b_3]'

The General solution to A$\underline{x}$=$\underline{b}$

Types of solutions...

Three important numbers for any matrix, m,n,r

Several types of problems

1) Full Column Rank (r=n):
all columns linearly independent
N(A)=Z
Either one or none solution

2) Full Row Rank (r=m):
no zero rows
C(A) = R^m
N(A) always non-trivial if n>m
At least one solution, usually infinite solutions

3) Full Row and Column Rank (r=n=m): Invertible square matrices: exactly 1 unique solution.

The BIG Ideas

Four critical ideas required to understand Vector Spaces and Subspaces

1) *Linear Independence*

2) *Span*

3) *Basis*

4) *Dimension*

Linear Independence

Definition:

Given a set of vectors, $\underline{a}_1,\underline{a}_2,...,\underline{a}_n$ and scalars $x_1,x_2,...,x_n$ the vectors are said to be "*Linearly Independent*" if the only linear combination that adds to zero i.e.

$x_1\underline{a}_1+x_2\underline{a}_2+x_3\underline{a}_3+...+x_n\underline{a}_n = \underline{0}$ (1)

is when

$x_1=x_2=x_3=...=x_n=0$

Comments:

1) a set of vectors is either Linearly Independent or Linearly Dependent

(not e.g. $\underline{a}_3$ is dependent on the rest...)

2) if $\underline{a}_1,\underline{a}_2,...,\underline{a}_n$ are real vectors in R^m, then we already have an algorithm to determine if a set of vectors is independent or not!

A) Eq. (1) is equivalent to ________ where

B) Therefore:

Linear Independence

Example:

$\underline{a}_1$ = [0 1 0]', $\underline{a}_2$= [1 0 1]', $\underline{a}_3$ = [1 -2 1]'

Span

Definition:

Given a set of vectors, $\underline{a}_1,\underline{a}_2,...,\underline{a}_n$
span(a1,a2,...,an) is the vector subspace formed by all linear combinations of the vectors

Comments:

we also say a set of vectors $\underline{a}_1,\underline{a}_2,...,\underline{a}_n$ *spans* some subspace S

Examples:
the columns of a matrix A span __________

the special solutions span _______________

Span

Examples:

Basis (Super Important idea!)

Definition:

A set of vectors, $\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_n$ forms a ***basis*** for a vector space (or subspace) S iff

1) they are linearly independent
2) they span S

Comments:

1) a basis is the *minimal* set of vectors required to reconstruct every vector in a vector space

Example: the pivot columns of A form a basis for _________

2) There are an *infinite* number of bases for any vector space S

Example: The columns of *any* ____________ 3x3 matrix A forms a basis for R^3

Basis: Important properties

Theorem:
given a particular basis for a vector space S any other vector $\underline{v}$ in S is described by a unique linear combination of the basis vectors.

Proof:

1) let $\underline{x}_1,\underline{x}_2,\ldots\underline{x}_n$ be a basis for a vector space S
2) let $\underline{v}$ be any other vector in S

Dimension of a vector space

Definition:
The dimension of a vector space S is the *number* of vectors in any basis for S

Comment:

While there are an infinite number of bases for any vector space S, the number of basis vectors (i.e. the dimension of S) is constant.

Example: Give 3 different bases for R^2

The dimension of R^2 is _______

Dimension of a vector space

Theorem:
Given a vector space S of dimension n, there are exactly n vectors in every basis for S.

Proof:

let $\underline{v}_1,\underline{v}_2,\ldots\underline{v}_m$ be one basis for a vector space S
let $\underline{w}_1,\underline{w}_2,\ldots\underline{w}_n$ be another basis S
show that _________

1) First assume n>m

Basis and Dimension of the 4 subspaces of a matrix A

Given A in R^{mxn} there are actually four fundamental subspaces associated with the matrix A

Name	Symbol	Subspace	Spanned by
1) Column Space			
2) Null Space			
3) Row Space			
4) Left Null Space			

Basis and Dimension of the 4 subspaces of a matrix A

Find the dimension and a basis for each of the 4 subspaces: a recipe

1) Reduce A to R=rref(A)

2) Get C(A) from R and A
basis = pivot columns of _____
dim(C(A)) = _____

3) Get N(A) from R
basis =
dim(N(A)) =

4) Get $C(A^T)$ from R
basis =
$\dim(C(A^T))$ = ____

5) Get $N(A^T)$ the hard-way (find Null space of pivot columns of A)
$\dim(N(A^T))$=

Basis and Dimension of the 4 subspaces of a matrix A

Example:

A= R=