Basis and Dimension of the 4 subspaces of a matrix A

Example:

Given A in \( \mathbb{R}^{m \times n} \) there are four fundamental subspaces associated with the matrix A:

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Subspace</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) Column Space</td>
<td>( C(A) )</td>
<td>Controls ( \mathbb{R}^m ) of solutions to ( Ax = b )</td>
<td>( \mathbb{R}^m )</td>
</tr>
<tr>
<td>2) Null Space</td>
<td>( N(A) )</td>
<td>( \mathbb{R}^n )</td>
<td>( \mathbb{R}^n )</td>
</tr>
<tr>
<td>3) Row Space</td>
<td>( C(A^T) )</td>
<td>Controls ( \mathbb{R}^n ) of solutions to ( A^T x = 0 )</td>
<td>( \mathbb{R}^n )</td>
</tr>
<tr>
<td>4) Left Null Space</td>
<td>( N(A^T) )</td>
<td>Controls ( \mathbb{R}^m ) of solutions to ( A^T y = 0 )</td>
<td>( \mathbb{R}^m )</td>
</tr>
</tbody>
</table>

Comments: \( C(A) \subseteq C(R) \)
Basis and Dimension of the 4 subspaces of a matrix A

Example:

\[ A = \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 1 & 3 \\ 2 & 4 & 6 & -2 & -2 \\ 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

LR = \[ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \]

The Row Space \( C(A^T) \): Spanned by the rows of \( A \) (i.e. columns of \( A^T \))

Subspace of

Dimension:

Basis:

Comments: \( C(A^T) \subseteq C(R^T) \)

Basis and Dimension of the 4 subspaces of a matrix A

Example:

\[ A = \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 1 & 3 \\ 2 & 4 & 6 & -2 & -2 \\ 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

LR = \[ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \]

The Left Null Space \( N(A^T) \): spanned by all solutions of \( A^T y = 0 \) - or - (Combinations of ______ of \( A \) that cancel to zero)

Subspace of

Dimension:

Basis:

Comments: \( N(A^T) \subseteq N(R^T) \)

Basis and Dimension of the 4 subspaces of a matrix A

Find the dimension and a basis for each of the 4 subspaces: a recipe

1) Reduce \( A \) to \( \text{R}=\text{ref}(A) \)

2) Column Space: Get \( C(A) \) from \( R \) and \( A \)
   \( \text{dim } C(A) = \), basis: _______________________

3) Null Space: Get \( N(A) \) from \( R \)
   \( \text{dim } N(A) = \), basis: _______________________

4) Row Space: Get \( C(A^T) \) from \( R \)
   \( \text{dim } C(A^T) = \), basis: _______________________

5) Left Null Space: Get \( N(A^T) \) the hard-way
   \( \text{dim } N(A^T) = \), basis: _______________________

Last Example: a rank-1 matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \]

What is the relationship between the 4 subspaces?
Orthogonality of the 4 subspaces

Point: For every matrix $A$ there are 4 fundamental subspaces

Two in $\mathbb{R}^n$:

Two in $\mathbb{R}^m$:

Moreover: these subspaces are "orthogonal complements" such that

Orthogonal Subspaces: Orthogonal Complements

Definition:

Two subspaces $S_1$ and $S_2$ in $\mathbb{R}^k$ are "orthogonal complements" if

1) all vectors in $S_1$ are orthogonal to those in $S_2$ (i.e. the two subspaces only share the 0 vector)

2) $\dim(S_1) + \dim(S_2) = k$

Comment: any basis from $S_1$ together with any basis from $S_2$ form a complete basis for $\mathbb{R}^k$, i.e. all vectors in $\mathbb{R}^k$ can be decomposed uniquely into a part in $S_1$ and $S_2$

Example:

$A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

Orthogonality of the 4 subspaces: The Big (Strangian) Picture
The Big Picture of $Ax=b$

Full row Rank:

The Big Picture of $Ax=b$

Full Column Rank:

The Big Picture of $Ax=b$

Full row and column Rank: Invertible Matrices