Lecture 17:
EigenProblems: Diagonalization and Matrix powers

Outline:
1) One more example (a rank one matrix)
2) $A\mathbf{x} = \lambda \mathbf{x}$ as $AS = S\Lambda$
3) Diagonalization and Factorization $A = S\Lambda S^{-1}$ (or $\Lambda = S^{-1}AS$)
4) Application of Diagonalization #1: Matrix Powers $A^n$
5) Requirements for Diagonalization: distinct eigenvalues
6) Application of Matrix Powers: Iterative Maps $\mathbf{u}_{k+1} = A\mathbf{u}_k$
   A) Fibonacci numbers
   B) Iterative methods for solving $A\mathbf{x} = \mathbf{b}$

EigenValues and Eigenvectors
Diagonalization and Factorization

1) All Eigenproblems can be written in matrix form as $AS = \Lambda S$

Definition: A matrix can be diagonalized if it has $n$ linearly independent eigenvectors i.e.

$A = \underline{\underline{\text{-------}}} \quad \text{or} \quad \Lambda = \underline{\underline{\text{-------}}}$

Mechanics of finding Eigenvalues and Eigenvectors

One More Example: $A = [1 \ 1 \ 1; 1 \ 1 \ 1; 1 \ 1 \ 1]$

Eigenvalues and Eigenvectors
Diagonalizable matrices have nice properties

1) Easy to prove the general checks for $|A|$ and $\text{Tr}(A)$

2) Gives a simple formula for Matrix Powers $A^n$
Diagonalization:
1) Unfortunately, not all matrices can be diagonalized:
   Example: \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \)

Theorem: all matrices with distinct eigenvalues can be diagonalized

Proof: (sketch for 3x3 case)
   Consider a system with 3 eigenvectors \( x_1, x_2, x_3 \) with corresponding distinct eigenvalues
   show that \( x_1, x_2, x_3 \) are linearly independent

Diagonalization: The Rules
1) If a matrix \( A \) has \( n \) distinct eigenvalues (no repeats), it will have \( n \) linearly independent eigenvectors and can always be diagonalized.
2) If a matrix has repeated eigenvalues it might be diagonalizable.
   Need to check the eigenvectors:
   A) if \( n \) linearly independent: diagonalizable
   B) if repeated eigenvectors (not linearly independent): not diagonalizable
3) If a matrix is symmetric (\( A^T = A \)): always diagonalizable (and more)

Applications of Matrix Powers: Iterative maps
Consider the iterative map \( u_{k+1} = Au_k \)
Applications of Diagonalization: Iterative maps

An alternative interpretation of the general solution: change of basis

Eigenvalues and Eigenvectors

Applications of Iterative Maps:

Example 1: Fibonacci Numbers

fibonacci sequence 0,1,1,__________________

Recursion relationship $f_{n+1} = f_n + f_{n-1}$

Question: what is the ratio of $f_{n+1}/f_n$ in the limit $n \to \infty$?

Answer: A cute linear algebra trick
Eigenvalues and Eigenvectors
Applications of Iterative maps

Example #2: Iterative methods for solving $Ax=b$