Lecture 20:
Eigenvalues and Eigenvectors of Symmetric Matrices

Outline:
1) Big Idea: If $A^T = A$ then $A = Q\Lambda Q^T$ with $\Lambda$ real always
   A) 3x3 example
   B) general 2x2 real eigenvalues
   C) general proof for all symmetric matrices
2) Properties of Symmetric matrices
   A) Spectral theorem
   B) sign of Pivots = signs of eigenvalues
3) Positive Definite Matrices (Symmetric, all eigenvalues >0)
   A) Tests
   B) Important PD matrices $A^T A$ and $AA^T$
4) Similar Matrices and Jordan Canonical form

Real Symmetric Matrices

Critical Properties of Eigenvalues and Eigenvectors:
If $A^T = A$

1) All eigenvalues are real
2) All eigenvectors can be chosen orthonormal
3) All symmetric matrices can be diagonalized
4) $A = SAS^{-1}$ becomes $A = Q\Lambda Q^T$

Real Symmetric Matrices

Example #1: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Real Symmetric Matrices

Example #2: $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ (General symmetric 2x2 matrix)
Preliminaries: A useful identity for proving general properties of Eigenvalues and eigenvectors of Real Symmetric Matrices

Given $A$ in $\mathbb{R}^{nxn}$ and $u, v$ in $\mathbb{R}^n$

**Theorem:** Given $A^T = A$, all the eigenvectors of $A$ with distinct eigenvalues are orthogonal (and can be chosen orthonormal)

**Proof:** Use the identity with $u = x_1$, $v = x_2$ where $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$

---

**Spectral Theorem for Symmetric Matrices**

All real symmetric matrices can be factored as $A = \ldots$

Alternative Interpretation: All symmetric matrices can be written as a sum of rank-one projection matrices
Spectral Theorem for Symmetric Matrices

Example: \( A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \)

Relationship between pivots and eigenvalues for Symmetric Matrices

In general Pivots (from elimination) are not related to eigenvalues...although

1) \(|A| = \_\_\_\_ = \_\_\_\_

2) For triangular matrices \_\_\_\_

3) For Symmetric Matrices: the signs of the pivots are the same as the signs of the eigenvalues
   
   i.e. \# of positive (negative) pivots = \# of positive (negative) eigenvalues

Relationship between pivots and eigenvalues for Symmetric Matrices

Proof: \( A^T = A \) then \# of positive (negative) pivots = \# of positive (negative) eigenvalues

1) LU decomposition for symmetric matrices

Counter Example (non-symmetric) \( A = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix} \)
Definitions:

A is a positive definite (PD) matrix if $A^T = A$ and all its eigenvalues are $> 0$.

If $\lambda \geq 0$, $A$ is said to be semi-positive definite.

Quick Tests for PD:

1) All the pivots are positive
   (Cholesky Factorization...)

2) The quadratic form $f(x) = x^T A x > 0$ for all $x \neq 0$

Two Extremely important PD matrices (needed for SVD)

Show that $A^T A$ and $AA^T$ are both at least semi-PD