Lecture 20:
Eigenvalues and Eigenvectors of *Symmetric* Matrices

Outline:

1) Big Idea: If $A^T=A$ then $A=Q\Lambda Q^T$ with Λ real always
   A) 3x3 example
   B) general 2x2 real eigenvalues
   C) general proof for all symmetric matrices

2) Properties of Symmetric matrices
   A) Spectral theorem
   B) sign of Pivots = signs of eigenvalues

3) Positive Definite Matrices (Symmetric, all eigenvalues >0)
   A) Tests
   B) Important PD matrices A^TA and AA^T

4) Similar Matrices and Jordan Canonical form

Real Symmetric Matrices

Critical Properties of Eigenvalues and Eigenvectors:

If $A^T=A$

1) All eigenvalues are real

2) All eigenvectors can be chosen orthonormal

3) All symmetric matrices can be diagonalized

4) $A=S\Lambda S^{-1}$ becomes $A=Q\Lambda Q^T$

Real Symmetric Matrices

Example #1: A=[1 0 1; 0 2 0 ; 1 0 1]

Real Symmetric Matrices

Example #2: A=[a b ; b d] (General symmetric 2x2 matrix)

Eigenvalues and Eigenvectors of Real Symmetric Matrices

Preliminaries: A useful identity for proving general properties of Eigenvalues and eigenvectors of Real Symmetric Matrices

given A in R^{nxn} and $\underline{u},\underline{v}$ in R^n...

Eigenvalues and Eigenvectors of Real Symmetric Matrices

Theorem: given A^T=A, all the eigenvectors of A with distinct eigenvalues are orthogonal (and can be chosen orthonormal)

Proof: use the identity with $\underline{u}=\underline{x}_1$ $\underline{v}=\underline{x}_2$ where $A\underline{x}_1=\lambda_1\underline{x}_1$ $A\underline{x}_2=\lambda_2\underline{x}_2$

Eigenvalues and Eigenvectors of Real Symmetric Matrices

Theorem: given real A^T=A, all the eigenvalues of A are real

Proof: use the identity using $\underline{u}=\underline{x}$ ($A\underline{x}=\lambda\underline{x}$) and $\underline{v}=\bar{\underline{x}}$ (complex conjugate of $\underline{x}$)

Preliminaries: Basic properties of complex numbers

Spectral Theorem for Symmetric Matrices

All real Symmetric matrices can be factored as A=__________

Alternative Interpretation: All symmetric matrices can be written as a sum of rank-one projection matrices

Spectral Theorem for Symmetric Matrices

Example: A=[1 3 ; 3 1];

Relationship between pivots and eigenvalues for Symmetric Matrices

In general Pivots (from elimination) are not related to eigenvalues...although

1) |A|=_________=____________

2) For triangular matrices ________________

3) For Symmetric Matrices: the signs of the pivots are the same as the signs of the eigenvalues

i.e. # of positive (negative) pivots= # of positive (negative) eigenvalues

Relationship between pivots and eigenvalues for Symmetric Matrices

Example: A=[1 3 ; 3 1]

Counter Example (non-symmetric) A=[1 6; -1 -4];

Relationship between pivots and eigenvalues for Symmetric Matrices

Proof: if A^T=A then # of positive (negative) pivots = # of positive (negative) eigenvalues

1) LU decomposition for symmetric matrices

Positive Definite Matrices

Definitions:

A is a positive definite (PD) matrix if A^T=A and all its eigenvalues are > 0.

If λ>= 0, A is said to be semi-positive definite.

Positive Definite Matrices

Quick Tests for PD:

1) All the pivots are positive

(Cholesky Factorization...)

Positive Definite Matrices

Quick Tests for PD:

2) The quadratic form $f(\underline{x})=\underline{x}^TA\underline{x} > 0$ for all $\underline{x} \neq 0$

Positive Definite Matrices

Two Extremely important PD matrices (needed for SVD)

Show that A^TA and AA^T are both at least semi-PD