Lecture 21:
Towards the SVD (singular Value Decomposition)

Outline:

1) Properties of Symmetric Positive Definite Matrices (all eigenvalues >0)
   A) Tests
   B) Important PD matrices A^TA and AA^T

2) Overview of Eigenvalue factorizations
   $AS=S\Lambda$ (generally)
   $A=S\Lambda S^{-1}$ (diagonalizable)
   $A=MJM^{-1}$ (non-diagonalizable)
   $A=Q\Lambda Q^T$ (symmetric)

3) The final Factorization, the incredible SVD
   Definition: $A=U\Sigma V^T$ (or $AV=U\Sigma$)
   Mechanics:
   The Big Picture:
   The Applications:
   Total Least squares, image compression, EOF analysis

Quick Review: Eigenvalues and Eigenvectors of Real Symmetric matrices

If A is real, square and symmetric:

All Eigenvalues of A are _______________

All Eigenvectors of A can be chosen _________________

All symmetric matrices can be diagonalized

Factorization: for $A^T=A$, A=____________

Positive Definite Matrices

Definitions:

A is a positive definite (PD) matrix if $A^T=A$ and all its eigenvalues are > 0.

If $\lambda >= 0$, A is said to be semi-positive definite.

Examples: A=[1 2 ; 2 1], [1 2 ; 2 4], [1 2 ; 2 5]

Positive Definite Matrices

Quick Tests for PD:

1) All the pivots are positive A = [1 2 ; 2 d]

(Cholesky Factorization...)

Positive Definite Matrices

Quick Tests for PD:

2) The quadratic form $f(\underline{x})=\underline{x}^T A\underline{x} > 0$ for all $\underline{x} \neq 0$

Positive Definite Matrices

Two Extremely important PD matrices (needed for SVD)

Show that A^TA and AA^T are both at least semi-PD

Summary of Eigenvalue/Eigenvector Factorizations

General Square A: ____________

diagonalizable A = ____________

non-Diagonalizable A = _____________

Symmetric Square A: Always Diagonalizable

Eigenvalues are _____________
Eigenvectors can be chosen _______________
Factorization: A=__________

Positive Definite Matrices: __________________
Positive Semi-Definite Matrices:______________

Onward to the SVD

Definition: Every matrix A (even non-square mxn) can be factored into its Singular Value Decomposition (SVD)

$$A=U\Sigma V^T$$

where U, and V are orthonormal matrices $U^TU=I$, $V^TV=I$

and Σ is a diagonal matrix

where

The incredible SVD: some comments

1) We need to understand where these three matrices come from, and how to compute them (and what they really mean);

2) However, to begin to see the utility of the SVD, assume A is invertible,

then if A=__________ A^{-1} =___________

3) More amazing...if A isn't even square, we can still use the SVD to define the pseudo-inverse A^{+} such that

$\underline{x}=A^{+}\underline{b}$ is the shortest-least squares solution to $A\underline{x}=\underline{b}$

The incredible SVD: some comments

4) An alternative way to view the SVD

Computing the SVD

Let $A=U\Sigma V^T$ (and nxn for the moment)

Then A^TA =

And AA^T =

But, these matrices have two important properties

1)___________________ 2)______________

Computing the SVD

Let $A=U\Sigma V^T$ (and nxn for the moment)

Then A^TA =

And AA^T =

Symmetry implies:

And PSD implies:

Therefore:

V contains__________________________

U contains__________________________

and Σ contains__________________

Computing the SVD: a recipe

Computing the SVD: a recipe

Example: A=[2 2 ; -1 1] (square invertible matrix)

Computing the SVD: a recipe

Example: A=[3 3 ; 4 4] (square singular matrix)