Lecture 23:
Applications of the SVD

Outline:
1) Quick Review
2) Application 1: Total Least Squares
   A recipe
   Interpretation
3) Application 2: Pattern Recognition and Empirical Orthogonal
   Functions (EOF's/PCA etc.)
   Examples: Image reconstruction and compression
   Classification of 1-D Topography (EOF analysis)

Quick Review

SVD of a general mxn matrix A with rank r

\[
\begin{bmatrix}
A
\end{bmatrix} = \begin{bmatrix}
U & \Sigma & V^T
\end{bmatrix}
\]

Matlab: \([U,S,V]=svd(A)\)

SVD: the economy sized version

SVD of a general mxn matrix A with rank r

\[
\begin{bmatrix}
A
\end{bmatrix} = \begin{bmatrix}
U & \Sigma & V^T
\end{bmatrix}
\]

Matlab: \([U,S,V]=svd(A,0)\)

Least squares problems using the SVD

if A is mxn overdetermined then the solution \(\hat{x}\) that minimizes

\[||e||=||b-A\hat{x}||\]

is \(\hat{x} = x^+ = A^+b\) where \(A^+ = V\Sigma^+U^T\) is the pseudo-inverse

Comments:
1) if A is invertible \(A^+ = \) ______ and \(||e|| = \) ______
2) if A is full column rank \(\hat{x}^+\) is the unique solution to
   ______
3) in general \(g = AA^+b\) is the projection of \(b\) onto ______
Least squares problems using the SVD

Total Least squares problems using the SVD

A recipe (in matlab):

Total Least squares problems using the SVD

Total Least squares problems using the SVD
Total Least squares problems using the SVD

A recipe (in matlab):

\[
x = [x_1 \ x_2 \ \ldots \ x_m]; \quad y = [y_1 \ y_2 \ \ldots \ y_m];
\]
\[
M = [x \ y];
\]
\[
xMean = mean(M);
\]
\[
M = M - ones(size(x))*xMean;
\]
\[
[U,S,V] = svd(M);
\]
\[
u = V(:,2);
\]
\[
v = V(:,1);
\]
\[
\text{line} = @(t) ones(size(t))*xMean + t*v';
\]
\[
X = \text{line}(t);
\]
\[
\text{plot}(X(:,1),X(:,2));
\]

A demo

Application #2: Pattern recognition and image compression using the SVD

![Image](http://www.metmuseum.org/toah/hd/durr/hod_43.106.1.htm)

Durer's icon of Melancholy portrays the dangers of obsessive study. Note the many symbols of mathematics and alchemy.

(Princeton, History 291 Web page)

The Big Picture

Image Compression

Given an original 359 x 371 Image:

We can write it as 359 x 371 matrix A with SVD \( A = U \Sigma V^T \)

where \( U \) is 359 x 359, \( \Sigma \) is 359 x 371 and \( V \) is 371 x 371
Image Compression

The spectral theorem for the SVD says that the matrix $A$ can also be written as a sum of rank-1 matrices

$$A = U \Sigma V^T$$

where each rank-1 matrix $u_i v_i^T$ is the size of the original image.

Comments:
1) Each of these matrices is a mode.
2) The strength of each mode is given by the singular value $\sigma_i$
3) Because the singular values are rank ordered $\sigma_1 \geq \sigma_2 \geq \sigma_3 \ldots \geq \sigma_r \geq 0$

significant compression of the image is possible if the spectrum of singular values has only a few large values

Example: Storage for full Matrix is $359 \times 371 = 133189$ pixels
Storage for a single mode is $359 \times 371 + 1 = 731$ pixels
Compression for a single mode is $99.45\%$

Spectrum of singular values for $A$

Here the spectrum is principally contained in the first ~200 modes

Mode reconstruction using Matlab

$$[U, S, V] = \text{svd}(A)$$
$$B = U(:, 1) * S(1, 1) * V(:, 1)'$$

Mode reconstruction with 1 modes
Or as a sum of the first 10 modes as

\[ B = U(:,1:10) \cdot S(1:10,1:10) \cdot V(:,1:10)' \]

which only uses 5% of the storage \((10 \times 359 + 10 \times 371 + 10 = 7310 \text{ pixels})\) vs \(350 \times 371 = 133189 \text{ pixels}\).

**Application #3: Pattern Extraction -- a real world research example of EOFs**

*A Global Analysis Of Mid-ocean Ridge Axial Topography*

Chris Small (LDEO)

*GEOPHYSICAL JOURNAL INTERNATIONAL* 116 (1): 64-84 JAN 1994
The data: cross axis topography profiles from different spreading rates

![Graph](image1.png)

Now the rows of $V^T$ form an orthonormal basis for the row space of $A$

i.e. each profile (row of $A$) can be written as a linear combination of the rows of $V^T$ or

$$A = CV^T$$

Inspection of the SVD shows that $C = U\Sigma$.

Here, the rows of $V^T$ are known as **Empirical Orthogonal Functions** or EOFs.

EOF analysis

If the spectrum of Singular values contains a few large values and a long tail of very small values, it may be possible to reconstruct the rows of $A$ with only a small number of EOFs. The spectrum for this data looks like

![Graph](image2.png)

Here much of the information is included in the first 4 EOF's.
**EOF analysis**

If the spectrum of Singular values contains a few large values and a long tail of very small values, it may be possible to reconstruct the rows of A with only a small number of EOFs. The spectrum for this data looks like

Here much of the information is included in the first 4 EOF’s.

**Intermediate spreading rate**

And we can reconstruct individual profiles as combinations of the first 4 EOF’s. For example here is one for a slow spreading rate.
Fast spreading rate

EOF reconstruction, sample 100=(−617.3,136.3,−557.2,135.8)