Lecture 23:
Applications of the SVD

Outline:

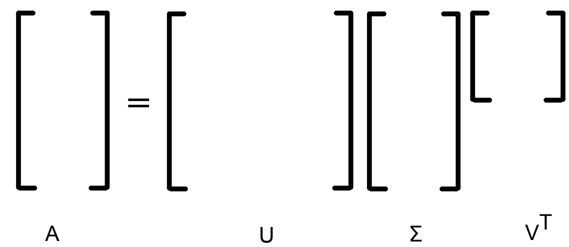
1) Quick Review

2) Application 1: Total Least Squares
   A recipe
   Interpretation

3) Application 2: Pattern Recognition and Empirical Orthogonal Functions (EOF's/PCA etc.)
   Examples: Image reconstruction and compression
   Classification of 1-D Topography (EOF analysis)

Quick Review

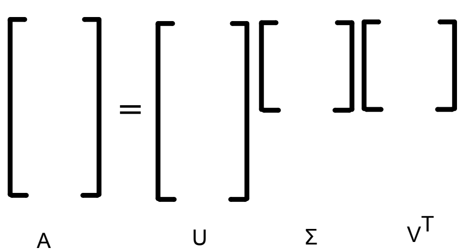
SVD of a general mxn matrix A with rank r



Matlab: [U,S,V]=svd(A)

SVD: the economy sized version

SVD of a general mxn matrix A with rank r



Matlab: [U,S,V]=svd(A,0)

Least squares problems using the SVD

if A is mxn overdetermined then the solution $\hat{\underline{x}}$ that minimizes

$$||\underline{e}||=||\underline{b}-A\hat{\underline{x}}||$$

is $\hat{\underline{x}}=\underline{x}^+=A^+\underline{b}$ where $A^+=V\Sigma^+U^T$ is the pseudo-inverse

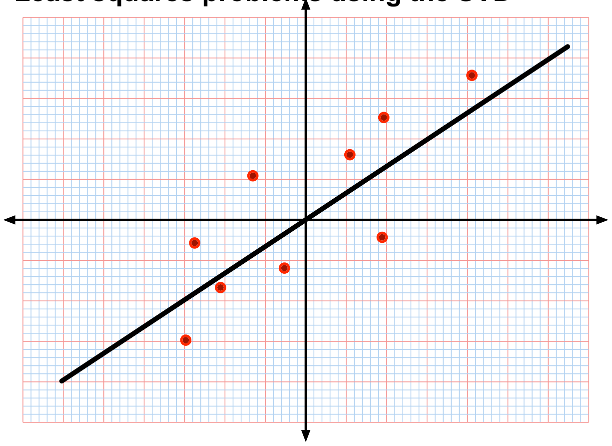
Comments:

1) if A is invertible A^+=______ and $||\underline{e}||$=_______

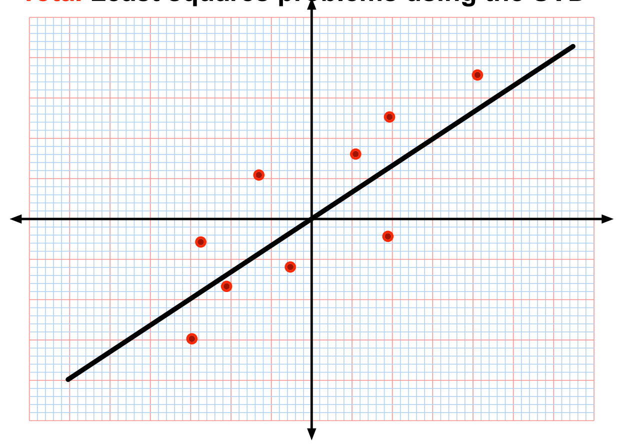
2) if A is full column rank $\underline{x}^+$ is the unique solution to
____

3) in general $\underline{p}=AA^+\underline{b}$ is the projection of $\underline{b}$ onto _______

Least squares problems using the SVD



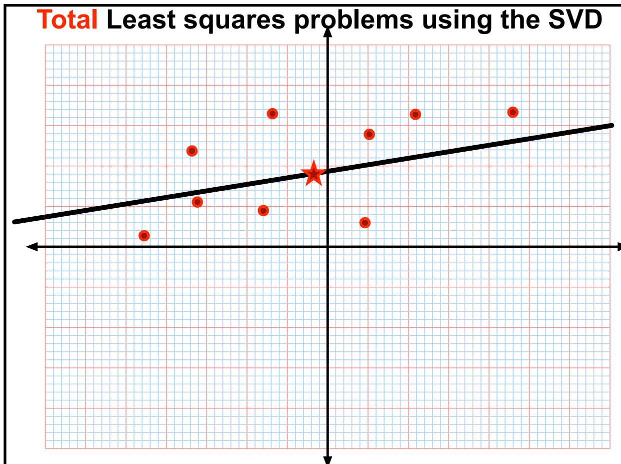
Total Least squares problems using the SVD



Total Least squares problems using the SVD

A recipe (in matlab):

Total Least squares problems using the SVD



Total Least squares problems using the SVD

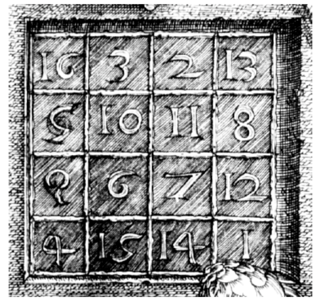
A recipe (in matlab):

```
x=[ x1 x2 ... xm]';   y = [ y1 y2 ... ym ]';  % set x, y data
M = [ x y];                                   % set Matrix
xMean = mean(M);                              % center of mass of data
M = M - ones(size(x))*xMean;                  % de-mean data
[ U,S,V] = svd(M);                            % get SVD of data
u = V(:,2);                                   % error vector
v = V(:,1);                                   % basis for line l(t)

% fancy stuff for plotting etc
line = @(t) ones(size(t))*xMean + t*v';
X = line(t);
plot(X(:,1),X(:,2));
```

A demo

Application #2: Pattern recognition and image compression using the SVD



Detail from Durer's "Melencholia I", 1514
(359 x 371 pixel image)

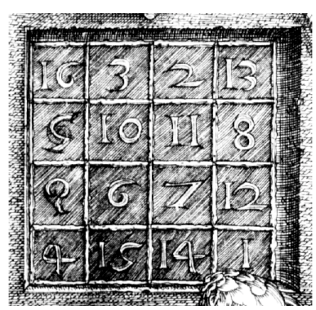
The Big Picture

Durer's icon of Melancholy portrays the dangers of obsessive study. Note the many symbols of mathematics and alchemy. (Princeton, History 291 Web page)

http://www.metmuseum.org/toah/hd/durr/hod_43.106.1.htm

Image Compression

Given an original 359 x 371 Image:



We can write it as 359 x 371 matrix A with SVD $A=U\Sigma V^T$

where U is 359 x 359, Σ is 359 x 371 and V is 371 x 371

Image Compression

The spectral theorem for the SVD says that the matrix A can also be written as a sum of rank-1 matrices

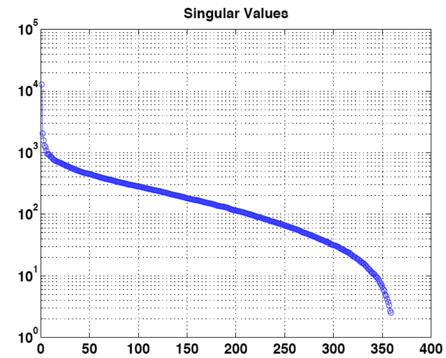
$A = U\Sigma V^T =$

where each rank-1 matrix $u_i v_i^T$ is the size of the original image.

Comments:
1) Each of these matrices is a *mode*.
2) The strength of each mode is given by the singular value σ_i
3) Because the singular values are rank ordered $\sigma_1 \geq \sigma_2 \geq \sigma_3 \ldots \geq \sigma_r \geq 0$ significant compression of the image is possible if the spectrum of singular values has only a few large values

Example: Storage for full Matrix is 359x371=133189 pixels
Storage for a single mode is 359 + 371 +1= 731 pixels
Compression for a single mode is 99.45%

Spectrum of singular values for A



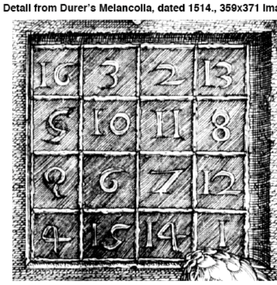
Here the spectrum is principally contained in the first ~200 modes

Mode reconstruction using Matlab

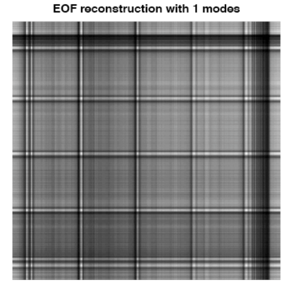
Mode reconstruction using Matlab

```
[U,S,V]=svd(A)
B=U(:,1)*S(1,1)*V(:,1)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



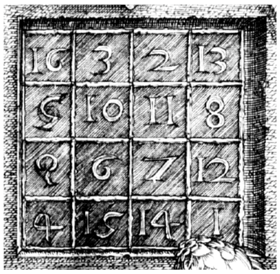
EOF reconstruction with 1 modes



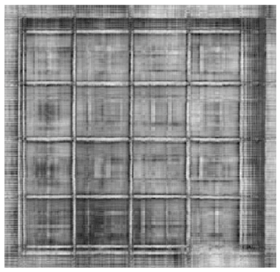
Or as a sum of the first 10 modes as

```
B=U(:,1:10)*S(1:10,1:10)*V(:,1:10)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



EOF reconstruction with 10 modes



which only uses 5% of the storage $(10 \times 359 + 10 \times 371 + 10 = 7310$ pixels vs $359 \times 371 = 133189$ pixels.

Matlab Demo

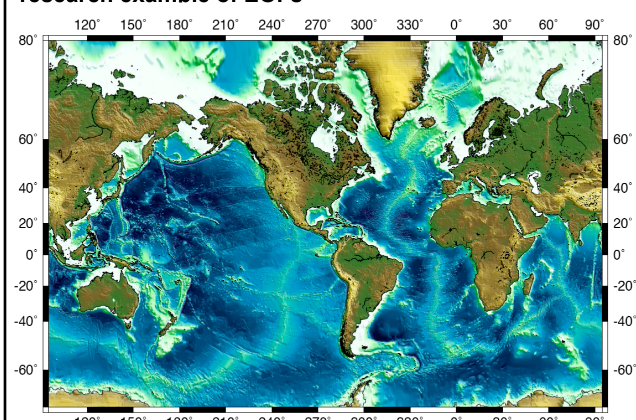
Application #3: Pattern Extraction -- a real world research example of EOFs

A Global Analysis Of Midocean Ridge Axial Topography

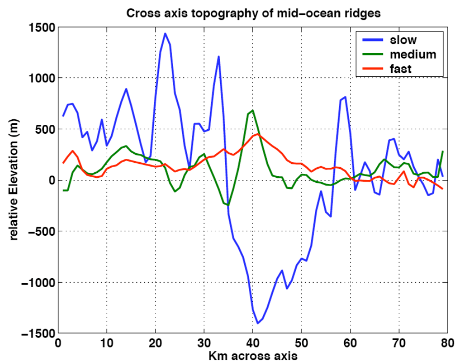
Chris Small (LDEO)

GEOPHYSICAL JOURNAL INTERNATIONAL 116 (1): 64-84 JAN 1994

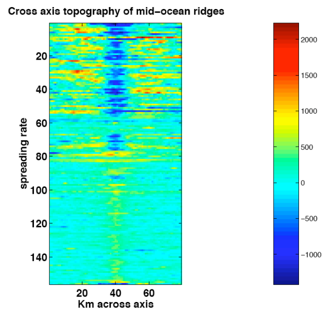
Application #3: Pattern Extraction -- a real world research example of EOFs



The data: cross axis topography profiles from different spreading rates



Form a matrix A (179×80) of elevation vs. distance across the ridge



and again take the SVD $A = U\Sigma V^T$. Here U is the same size as A and Σ and V are both square 80×80 matrices.

EOF analysis

Now the rows of V^T form an orthonormal basis for the row space of A

i.e. each profile (row of A) can be written as a linear combination of the rows of V^T or

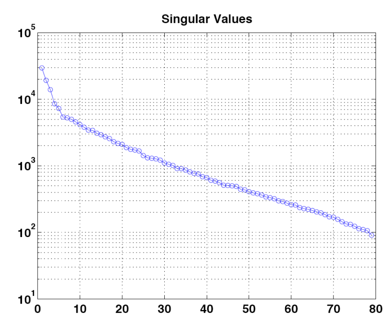
$$A = CV^T$$

Inspection of the SVD shows that $C = U\Sigma$.

Here, the rows of V^T are known as *Empirical Orthogonal Functions* or EOFs.

EOF analysis

If the spectrum of Singular values contains a few large values and a long tail of very small values, it may be possible to reconstruct the rows of A with only a small number of EOFs. The spectrum for this data looks like



Here much of the information is included in the first 4 EOF's

EOF analysis

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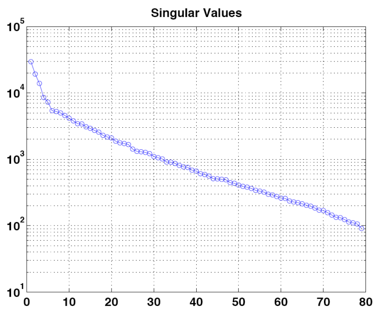

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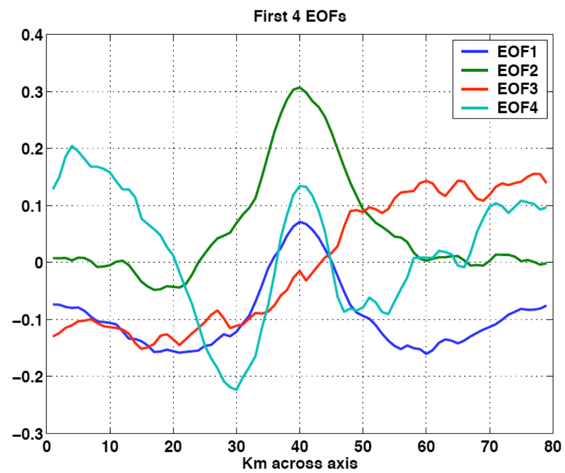
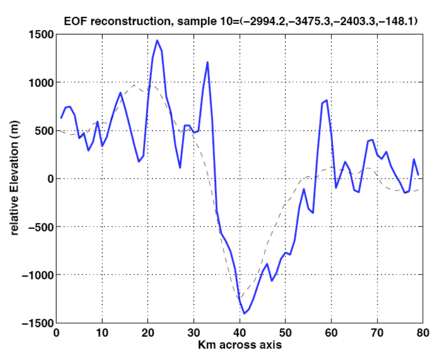
First 4 EOF's

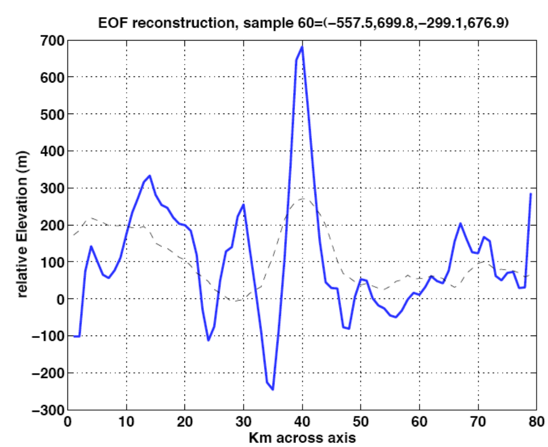


And we can reconstruct individual profiles as combinations of the first 4 EOF's.

For example here is one for a *slow* spreading rate



Intermediate spreading rate



Fast spreading rate