Lecture 16: Numerical Linear Algebra

Outline

0) Newton Demo for Linear systems:
1) Overview of Linear Algebra
2) Basics: Vector and Matrix Norms
3) The condition number of a matrix cond(A)
4) The condition number and error estimation in $Ax=b$

Newton's method for $F(x)=0$

Example: Linear problem
Overview of Linear Algebra

<table>
<thead>
<tr>
<th>subject:</th>
<th>Linear Systems</th>
<th>Least Squares</th>
<th>Eigen Problems</th>
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</thead>
<tbody>
<tr>
<td>Equations:</td>
<td>$A\mathbf{x} = \mathbf{b}$</td>
<td>$A^T A \mathbf{x} = A^T \mathbf{b}$</td>
<td>$A\mathbf{x} = \lambda \mathbf{x}$</td>
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<tr>
<td>Algorithms:</td>
<td>Elimination (Gauss, GJordan)</td>
<td>Gram-Schmidt</td>
<td>factor $P(\lambda) =</td>
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<td></td>
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<td>Householder</td>
<td>Find $N(A-\lambda I)$</td>
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<td>Factorizations:</td>
<td>$PA = LU$</td>
<td>$A = QR$</td>
<td>$AS = S\Lambda$</td>
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<td></td>
<td>$PA = LDL^T$</td>
<td></td>
<td>$A = S\Lambda S^{-1}$</td>
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<td></td>
<td>$A = CC^T$</td>
<td></td>
<td>$A = Q\Lambda Q^T$</td>
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<td>$A = U\Sigma V^T$</td>
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Outline of Lectures for Numerical Linear Algebra

1) Numerical issues and error analysis for NLA (this lecture)
2) Direct methods for solving square linear systems $A\mathbf{x} = \mathbf{b}$
   Gaussian Elimination with Partial Pivoting and the LU factorization
3) Direct methods for solving linear least squares problems
   Normal Equations, Orthogonalization and the QR factorization
4) Iterative methods for solving $A\mathbf{x} = \mathbf{b}$ and relationship to eigenvalue problems
   Simple iterative schemes
   Power method and variations for eigenvalues/eigenvectors
   Introduction to Krylov subspace schemes
Numerical Issues:

Floating Point Errors:
   Considerable amount of mults & adds can compound FP errors
   Need **Stable** schemes that don't amplify FP Errors

Ill-Conditioning:
   Possibility of "near-singular" matrices (badly behaved, particularly with FP errors)

Performance and scaling:
   Solution of large systems can be extremely expensive computationally
   e.g. $O(N^3)$ -- need fast algorithms for big problems

Quantifying Errors and **Condition Number** of a matrix:

Motivation: Need a scalar measure of how "singular" a matrix is

Why not the Determinant $|A|$?
Quantifying Errors and *Condition Number* of a matrix: Vector Norms

Definition: the $p$-norm of a vector is defined as $\|x\|_p$

Example: $x=[1\ -1\ 1]'$

1-norm: (Manhattan Distance)

2-norm: (Euclidean distance)

$\infty$-norm: (max(abs(x))

The unit "sphere" in the 1,2 and inf-norm in $\mathbb{R}^2$
Properties of Vector Norms

All vector norms have 3 fundamental properties

Matrix p-norms
"Induced" by vector p-norms

Definition: the p-norm of a matrix $\|A\|_p$

Point: describes the maximum distortion of the "unit-sphere"

Quick Definitions (and matlab):

$\|A\|_1 = \text{norm}(A,1) = \max(\text{sum(abs(A)))}$ (is max 1-norm of columns of A)

$\|A\|_2 = \text{norm}(A) = \max(\text{svd}(A))$ (maximum singular value of A)

$\|A\|_\infty = \text{norm}(A,\text{'inf'}) = \max(\text{sum(abs(A'))})$ (is max 1-norm of rows of A or $\|A^T\|_1$)
Matrix p-norms

Examples:
1) $\|I\| =$

2) $\|D\| =$

3) let $Q$ be orthonormal s.t. $Q^TQ = I$, then $\|Q\|_2 =$

4) $A =$

Properties of Matrix Norms

All Matrix norms are defined by 3 fundamental properties

In addition, matrix p-norms satisfy
Matrix p-norms

Proof: $\|A\|_1 = \max(\text{sum}(\text{abs}(A)))$ (is max 1-norm of the columns of A)

Matrix p-norms

Proof: $\|A\|_2 = \max(\text{singular value})$
The condition number of a matrix $A$

Definition: the condition number of a matrix $A$, $\text{cond}(A)$ (or sometimes $\kappa(A)$ ) is $\|A\| \|A^{-1}\|$

Condition number of special matrices

1) $\text{cond}(I) =$

2) $\text{cond}(D) =$

3) $\text{cond}(A)$ for $A$ singular:

4) $\text{cond}(\alpha A) =$

The condition number of a matrix $A$

Example: $A = \begin{bmatrix} (1+\alpha) & 1 \\ 1 & (1+\alpha) \end{bmatrix}$ \quad $\varepsilon_{\text{mach}} < \alpha << 1$
The condition number and error analysis of $Ax=b$

Relative forward and backwards error

The condition number and error analysis of $Ax=b$

Relative correction error and the residual for iterative methods