Lecture 24: Starting to put it all together #3...
More 2-Point Boundary value problems

Outline

1) Our basic example again:
   \[-u'' + u = f(x); \quad u(0)=\alpha, u(L)=\beta\]
2) Solution of 2-point Boundary value problems
   A) Shooting Methods
   B) Finite-Difference methods
   C) Higher-order: Spectral - Collocation methods
   D) Variation #2: Non-linear problems
      \[-u'' + u^p = f(x); \quad u(0)=\alpha, u(L)=\beta\]
   E) Galerkin Finite Elements

An Example problem: Simplest Steady-state 2-point boundary value problem
Solving 2-pt ODE Boundary Value Problems (BVPs)

Method #3: Spectral Collocation Schemes: Increasing accuracy

Big idea: express approximate continuous function \( \hat{u}(x) \) as a finite dimensional linear combination of \( n \) basis functions.

Collocation method: A review
A) choose \( n \) basis functions (e.g. Lagrange Polynomials)
B) choose \( n \) Collocation Points (e.g. N Chebyshev points)
C) require that the ODE is satisfied exactly at the \( N \) collocation points

D) this will reduce a linear problem to an equivalent \( nxn \) linear algebra problems \( Aw = f \), \( u(x) = Mw \) (for many systems \( A \) will be dense)

![FD vs Cheb BVP errors](image)
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Spectral Collocation Schemes: Pros and Cons

Pros:
- Increased accuracy for relatively small N
- Not much harder than low-order finite difference
  (still reduces to a linear algebra problem)

Cons:
- Less control of mesh points
  (for Chebyshev, different N have different points)
- Produce Dense matrices (Ax=b much more expensive)
- Can be problematic for non-linear problems

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Theme and variation #2: adding non-linearity

Let's change the radiation model to a generalized black-body radiation:

Method #1: Shooting --easy (but sensitive)
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Theme and variation #2: adding non-linearity

Shooting errors (some analysis)

Method #2: Non-linear finite difference: still replace differential operators with their discrete approximations, however the resulting equations are no longer linear
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Method #2: Non-linear finite difference: Newton's method

```matlab
function [F,J] = radiationModel(u,f,p,D)

% radiationModel - returns residual and Jacobian for Discrete approximation
% to the non-linear 2-point ODE BVP, for input to newton

% -u''+u^p = f(x) with dirichlet Boundary conditions
%
% [F,J] = radiationModel(u,f,p,D)
%
% u: current guess for solution (column vector)
% f: discrete vector for forcing function on rhs (including boundary condition)
% p: radiation exponent
% D: Second Derivate matrix (from either finite Difference or collocation methods)
%
% F: vector of residuals F(x)=-D*u+u^p-f
% J: Jacobian -D+diag(p*u.^(p-1))

N = length(u);

F = -D*u+u.^p-f;  \% residual
J = -D + spdiags([ p*u.^((p-1))],0,N,N); \%jacobian

% fix boundary conditions for dirichlet problem
F([1 end]) = 0;
J(1,:) = [1 zeros(1,N-1)];
J(end,:) = [zeros(1,N-1) 1];
```
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Method #2: Newton's method -- issues

1) Initial Guess:

2) Higher order Collocation methods

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Method #4: Finite Element Galerkin techniques (functional least-squares)

A) Like Collocation method, start out with expansion in basis functions

B) Unlike Collocation methods basis functions are local with "compact support"
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Method #4: Finite Element Galerkin techniques (functional least-squares)

Simplest 1-d Basis function: the "hat functions" basis for all piecewise linear functions on a line.
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Method #4: Finite Element Galerkin techniques (functional least-squares)

The equivalent view by "element"
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Method #4: Finite Element Galerkin techniques (functional least-squares)
Analogy with Linear Least Squares

Orthogonality of functions