Lecture 25:  Putting it all together #4...
Galerkin Finite Element Methods for ODE BVPs

Outline

1) Galerkin Finite Elements: the big ideas
2) A specific Example: Piecewise linear elements
   A) Elements and Basis functions
   B) Defining the residual
   C) Galerkin Finite Elements and Least-squares problems
   D) The "weak form" and \((K+M)u=f\)
3) Computational Issues and Algorithms

Galerkin FEM: the big ideas

Still solving our basic problem: \(-u''+u =f\) \(u(0)=\alpha\) \(u(L)=\beta\) or more generally

1) Like Collocation method, start out with expansion in basis functions

Unlike Collocation methods basis functions are local with "compact support"
Galerkin FEM: the big ideas

2) Like Collocation method, find discrete set of weights $w$ that satisfy some constraint on the continuous ODE

   A) e.g. for collocation, find $w$, st the ODE is satisfied exactly at the N collocation points

   B) For Galerkin Finite Elements, find $w$ to minimize the residual in a Least-Squares sense

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Galerkin Finite Elements

Example problem: Approximating $u(x)$ as a piecewise linear function
Galerkin Finite Elements

Example problem: Approximating $u(x)$ as a piecewise linear function
The "Element View"

![Diagram of Galerkin Finite Elements]

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Galerkin Finite Elements

Example problem: Approximating $u(x)$ as a piecewise linear function
As a sum of Basis functions (Hat Functions)

![Diagram of Galerkin Finite Elements]

Galerkin Finite Elements

Example problem: Approximating $u(x)$ as a piecewise linear function
As a sum of Basis functions (Hat Functions)

Point: by adjusting weights ($u$), we can construct all piecewise linear functions for the chosen set of nodes $x$.

Question: How do we choose $u$ to find the "best fit" piecewise linear function, that satisfies the ODE in a least-squares sense

Galerkin Finite Elements

Galerkin FEM as a least squares problem:

The Continuous problem: find $u(x)$ that satisfies

The Discrete problem: find $\bar{u}(x)$ that minimizes the residual i.e.

Point: the residual $r(x,u)$ is a continuous function, how to adjust $u$ to minimize some measure of $r$
Galerkin Finite Elements

Review of Linear Least squares in $\mathbb{R}^n$

Point: minimizing $r$ simply requires that $r$ is orthogonal to all the basis vectors $a_i$ (i.e. $a_i^T r = 0$ for all $i$).

Question: what is the equivalent statement for continuous $r(x,u)$?

Galerkin Finite Elements

The inner product for functions:

Definition: the inner-product of two functions $u, v$ on the interval $[a, b]$ is

Compare to the inner product of two vectors $u, v$ in $\mathbb{R}^n$ (the dot product)
Galerkin Finite Elements

Orthogonal Functions

Definition: two functions $u, v$ are said to be orthogonal on the interval $[a, b]$ if

For Galerkin Finite Elements: we require that the residual is orthogonal to all the basis functions, i.e.

Galerkin Finite Elements

Putting it together: Deriving the "weak form" equations
Galerkin Finite Elements

Tricks of the trade: from weak form to $A\mathbf{x} = \mathbf{b}$

Evaluating the integrals: The *stiffness matrix* $K$
Galerkin Finite Elements

Evaluating the integrals: The Mass Matrix M and the force vector f

Galerkin Finite Elements

Putting it together, the "element view": element matrices and global matrix assembly