Lecture 25: Putting it all together #4...
Galerkin Finite Element Methods for ODE BVPs

Outline

1) Galerkin Finite Elements: the big ideas
2) A specific Example: Piecewise linear elements
   A) Elements and Basis functions
   B) Defining the residual
   C) Galerkin Finite Elements and Least-squares problems
   D) The "weak form" and \((K+M)u=f\)
3) Computational Issues and Algorithms

Galerkin FEM: the big ideas

Still solving our basic problem: 
\[-u''+u = f \quad u(0)=\alpha \quad u(L)=\beta \]
or more generally

\[
\begin{align*}
\mathcal{L}u &= f \\
A u &= f \\
\mathcal{L} &= \frac{d^2}{dx^2} + 1
\end{align*}
\]

1) Like Collocation method, start out with expansion in basis functions

\[
\hat{u}(x) = \sum_{i=1}^{n} \phi_i(x) \omega_i
\]

Unlike Collocation methods basis functions are local with "compact support"
Galerkin FEM: the big ideas

2) Like Collocation method, find discrete set of weights $w$ that satisfy some constraint on the continuous ODE

A) e.g. for collocation, find $w$, st the ODE is satisfied exactly at the $N$ collocation points

B) For Galerkin Finite Elements, find $w$ to minimize the residual in a Least-Squares sense

$$\tilde{u}(x) = \sum_{j=1}^{n} q_j(x) \omega_j$$
$$\Gamma(x, \omega) = f(x) - L \tilde{u}$$
$$\Gamma = b - A \tilde{u}$$

Galerkin Finite Elements

Example problem: Approximating $u(x)$ as a piecewise linear function
Galerkin Finite Elements

Example problem: Approximating $u(x)$ as a piecewise linear function
The "Element View"

$u(x) = \sum_{i} \phi_i(x) u_i$

$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i} & x_{i-1} \leq x < x_i \\ 0 & \text{otherwise} \end{cases}$
Galerkin Finite Elements

Example problem: Approximating $u(x)$ as a piecewise linear function
As a sum of Basis functions (Hat Functions)

Point: by adjusting weights ($u$), we can construct all piecewise linear functions for the chosen set of nodes $x$.

Question: How do we choose $u$ to find the "best fit" piecewise linear function, that satisfies the ODE in a least-squares sense

Galerkin Finite Elements

Galerkin FEM as a least squares problem:

The Continuous problem: find $u(x)$ that satisfies

$$ Lu = f \quad L u(x) = u''(x) + u(x) = f $$

$u(0) = u(1) = 0$

The Discrete problem: find $\hat{u}(x)$ that minimizes the residual i.e.

$$ \hat{\text{r}}(x, u) = \sum_j \hat{r}_j(x) u_j $$

$$ r(x, u) = Lu - f = -\hat{u}'' + \hat{u} - f(x) $$

Point: the residual $r(x, u)$ is a continuous function, how to adjust $u$ to minimize some measure of $r$
Galerkin Finite Elements

Review of Linear Least squares in $\mathbb{R}^n$

Point: minimizing $r$ simply requires that $r$ is orthogonal to all the basis vectors $a_i$ (i.e. $a_i^T r = 0$ for all $i$).

Question: what is the equivalent statement for continuous $r(x,u)$?

Galerkin Finite Elements

The Inner product for functions:

Definition: the inner-product of two functions $u,v$ on the interval $[a,b]$ is

$$\langle u, v \rangle = \int_a^b uv \, dx$$

Compare to the inner product of two vectors $u,v$ in $\mathbb{R}^n$ (the dot product)

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\hat{b} = \sum_{i=1}^n u_i v_i$$
Galerkin Finite Elements

Orthogonal Functions

Definition: two functions \( u, v \) are said to be orthogonal on the interval \([a,b]\) if

\[
\langle u, v \rangle = 0
\]

For Galerkin Finite Elements: we require that the residual is orthogonal to all the basis functions, i.e.

\[
w_{\text{res}} = 0 \quad \text{for all } i \quad \langle e_i, r \rangle = 0 \quad A u = f
\]
Galerkin Finite Elements

Tricks of the trade: from weak form to $Ax=b$

Evaluating the integrals: The stiffness matrix $K$

$$\int_0^L a_i \partial_i^2 \partial_j^2 \, dx$$

Integration by parts:

$$\int a_i \partial_i \partial_j u \, dx = \left. a_i \partial_i u \partial_j \right|_0^L - \int_0^L a_i \partial_i (\partial_j u) \, dx$$

$$k_{ij} = \left. a_i \partial_i u \partial_j \right|_0^L + \int_0^L a_i \partial_i (\partial_j u) \, dx$$
Galerkin Finite Elements

Evaluating the integrals: The Mass Matrix $M$ and the force vector $f$

Putting it together, the "element view": element matrices and global matrix assembly