

Homework #4: Quadraphenia

Just 3 Questions: No major programming but I expect you to use Matlab to solve Questions 2 and 3. You are not required to submit code, however please show fundamental derivations required to solve these problems on your (computer/calculator/abacus) and answer the questions in the text.

1. Basic Quadrature rules. The Error Function $\text{erf}(x)$ is defined by the definite integral

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Given $\text{erf}(1) = 0.84270079294971$, calculate approximate values $Q \approx \text{erf}(1)$ and relative errors (i.e. $|\text{erf}(1) - Q|/\text{erf}(1)$) for each of the following methods (just write down the formulas and your calculated answers)

- (a) Mid-point
- (b) Trapezoidal
- (c) Simpson's Rule
- (d) 3-point Gaussian Quadrature (hint: use undetermined coefficients given the zeros of the legendre Polynomial $P_3(x) = (5x^3 - 3x)/2$. Be careful about the limits of integration).
- (e) **Extra Credit:** 4-point Gaussian Quadrature given $P_4(x) = (35x^4 - 30x^2 + 3)/8$ (hint: use Matlab `roots` and linear algebra functions to calculate weights and integration points).

2. A much harder function to integrate with single-interval rules is

$$f(x) = \frac{1}{1 + 100x^2}$$

Which has the integral

$$I(x) = \int_0^x f(t) dt = \tan^{-1}(10x)/10$$

If we try to evaluate $I(1) \approx 0.14711276743037$ by Simpson's rule, the relative error is 32%. Even a 5 Point Gauss-Legendre rule only gives $G_5 = 0.15113868537169$ with a relative error $\sim 2.7\%$.

Try to do better with extended Newton-Cotes formulas. Make a plot comparing the relative errors as a function of the number of function evaluations for these rules

- (a) Extended Trapezoidal Rule T_N for $N = 2^k$ panels ($2^k + 1$ points) where $k = 1 : 9$. (hint: use matlab's `linspace` and `trapz` functions and you can calculate $T_N(k)$ in about 5 lines)
- (b) Extended Simpsons Rule S_N : (hint: remember $S_N = \frac{4}{3}T_{2N} - \frac{1}{3}T_N$)
- (c) Extended Boole's Rule ($O(h^6)$)

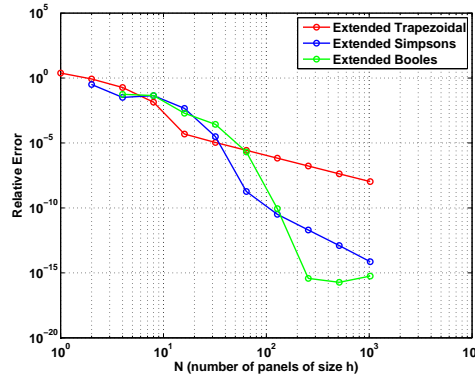


Figure 1: Comparison of Relative error between Extended Trapezoidal rule, Extended Simpson’s rule and Extended Boole’s Rule. Machine precision $\epsilon_{\text{mach}} \sim 10^{-16}$.

- (d) **Extra Credit:** Keep going to calculate the 8th order extended quadrature rule. Is this actually any more accurate than the Extended Boole’s rule? Explain your answer.

Figure 1 shows my best version of the plot for this problem (using all my tricks to maintain floating point accuracy). Note that Extended Boole’s rule can reduce the error to machine precision in about 256 function evaluations. **Estimate (or try to calculate)** the number of evaluations required by the Extended Trapezoidal Rule and Extended Simpson’s rule, to reach the same level of accuracy.

3. **The Great Boat Race:** You’re trying to set up a race course for rowing on a sinuous river with profile

$$f(x) = A \sin(kx)$$

where $A = 100\text{m}$, $k = 2\pi/300$ and x is in meters. If the starting line is at $x = 0$, where should you place the finish line ($x = L$) such that the actual race-course is 2000m long? (you can assume a very narrow river). Figure 2 shows a schematic of the set up. (Hints! (1) think arc-length. (2) **use matlab:** some useful functions are `cumtrapz`, `quad`, `fzero`. (3) Things can get tricky passing functions to functions to functions. Be careful and ask me if you get stuck)

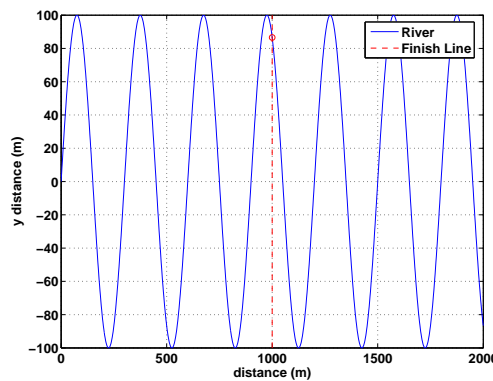


Figure 2: The Great Race schematic: Given a curvy river, figure out where to put the finish line (at $x = L$) so that the total race course is 2000m. The answer is *not* 1000m for this problem.