Homework #6: Numerical Linear Algebra

4 basic problems for numerical linear algebra

1. Quick definition check: Consider the matrices

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix} \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 3 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

   (a) define and find $||A||_1$ and $\mathrm{cond}_1(A)$, use Matlab to check your answer

   (b) Do the same for the infinity-norm ($||A||_\infty$, $\mathrm{cond}_\infty(A)$)

   (c) Use matlab to find the singular values of A and the $\mathrm{cond}_2(A)$

2. consider the problem $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1+\epsilon \\ 2 \end{bmatrix}$$

   where ϵ is a small parameter (to be determined below).

   (a) Is A ill or well conditioned? (e.g. find $\mathrm{cond}_1(A)$)

   (b) Find the LU decomposition of A (by hand)

      i. without partial pivoting

      ii. with partial pivoting

   (c) for $\epsilon = 10^{-2k}$ where $k = 1, \ldots, 10$, use matlab to calculate the solution to $A\mathbf{x} = \mathbf{b}$ *without* partial pivoting (use your result above), and **make a plot** of the relative error $r = ||\mathbf{x} - \mathbf{x}_{true}||/||\mathbf{x}_{true}||$ where $\mathbf{x}_{true} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is the exact solution. What is the largest value of k (or smallest value of ϵ) such that the solution is good to 6 digits of decimal precision?

   (d) repeat the problem but use matlab's backslash (i.e. `x=A\b`) which implements partial pivoting. What is the maximum relative error for this problem with pivoting?

3. Fun with the QR decomposition. Consider the over-determined problem $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

   Find the least squares solution to this problem using

   (a) The Normal Equations

   (b) QR factorization using modified Gram-Schmidt orthogonalization

   (c) QR Factorization using Householder transformations.

   To understand the algorithms, I'd like you to do these problems by hand (but you can use matlab to do the intermediate calculations and check your answers).

4. Quick application problem for least squares problems: Use any method you want (hint: backslash is nice) to find the best fit of the function $y = c_1 x + c_2 e^x$ through the (x, y) data points (1,2), (2,3), (3,5), (4,10), (5,15).

   (a) **Make a plot** comparing the best fit function to the data over the interval $x \in [0, 6]$.

   (b) What is the norm of the residual $||\mathbf{r}||_2$?

   (c) Do you think a straight line would be a better model? (This is a subjective question with no correct answer. Just justify your answer... and keep it brief).