Homework #7: Putting it all together

Two Questions with lots of options (and don’t forget page 2)

1. Consider the linear non-homogeneous 2-pt boundary value problem

\[ u'' + u = \sin 3x \quad 0 \leq x \leq L \quad u(0) = u(L) = 0 \]

which is related to the modes of vibration of a forced string of length \( L \).

(a) Find an analytic solution to this problem for general \( L \) (hint: remember your basic ODE’s for constant-coefficient problems). Is this solution unique? Explain your answer.

(b) Now solve the problem numerically for a string of length \( L = 10 \) using any one of the four methods: Shooting, 2nd-order finite difference, Chebyshev Collocation, or Galerkin Finite Elements. You can modify any of the demonstration codes or write your own. **Produce a representative plot of your solution compared to the analytic solution.**

(c) If we define the relative error as

\[ r = \frac{||u(x) - \tilde{u}(x)||_2}{||u(x)||_2} \]

where \( \tilde{u}(x) \) is your numerical solution evaluated at discrete points \( x \) and \( u(x) \) is the analytic solution evaluated at the same points, use your chosen method to find an approximate solution with \( r \leq 10^{-6} \) (if you can). Briefly discuss what you need to adjust to achieve this accuracy and why and **Present a plot showing \( r \) as a function of your adjustable parameter**

2. Non-linear fun: Now try to solve the related non-linear 2-point boundary value problem

\[ u'' + \sin(u) = \sin(3x) \quad 0 \leq x \leq 10 \quad u(0) = u(10) = 0 \]

using finite-difference or collocation methods plus Newton’s method.

(a) Solve the problem for an initial guess \( u_0 = 0 \). Use your results from problem 1 to choose accuracy conditions (how might you check if the solution is accurate?). **Produce a plot of your solution**

(b) Is your solution unique? Try starting newton from different initial conditions (and welcome to the wonderful world of non-linear ODE’s). For any solution, show that it satisfies at least the discrete ODE (i.e. plot \( u'' + u \) vs \( \sin(3x) \)). How many solutions can you find this way?

(c) **Extra credit:** Use shooting to systematically find solutions as a function of take-off angle. How many solutions can you find this way?

(d) **More Extra credit:** Compare both Finite Difference and Collocation methods for this problem? Does improved accuracy change things?

(e) **Extra Extra credit:** Can you explain this behavior?
For your convenience I have supplied a matlab routine

\[
[x, \text{ residual, nIterations }] = \text{newton}(@\text{func}, x0, \text{tol})
\]

in available on the web-site. This function takes as an argument a function with the interface \([F, J]=\text{func}(x)\), where \(F\) is the vector \(F(x)\) and \(J\) is the Jacobian evaluated at point \(x\). You can test this function by finding solutions that satisfy \(F(x) = 0\) to machine precision where

\[
\begin{align*}
F_1(x) &= (x_1 + 3)(x_2^3 - 7) + 18 \\
F_2(x) &= \sin(x_2 \exp(x_1) - 1)
\end{align*}
\]

Starting at \(x_0 = \begin{bmatrix} -0.5 & 1.4 \end{bmatrix}\).