Appendix A

Vector Calculus: a quick review

Selected Reading

H.M. Schey,. Div, Grad, Curl and all that: An informal Text on Vector Calculus, W.W. Norton and Co., (1973). (Good physical introduction to the subject)

Mase, George. Theory and problems of Continuum Mechanics: Schaum's outline Series. (Heavy on tensors but lots of worked problems)

Marsden, J.E. and Tromba, A.J. . Vector Calculus. W.H. Freeman (or any standard text on Vector calculus)

In modeling we are generally concerned with how physical properties change in space and time. Therefore we need a general mathematical description of both the variables of interest and their spatial and temporal variations. Vector calculus provides just that framework.

A.1 Basic concepts

Fields A field is a continuous function that returns a number (or sets of numbers) for every point in space and time $(\mathbf{x}, t)$. There are three basic flavours of fields we will deal with

scalar fields A scalar field $f(x, y, z, t)$ returns a single number for every point in space and time. Examples include temperature, salinity, porosity, density. . . .

vector fields A vector field $\mathbf{F}(x, y, z, t)$ returns a vector for every point in space and is readily visualized as a field of arrows. Examples include velocity, elastic displacements, electric or magnetic fields.

tensor fields A second rank tensor field $\mathsf{D}(x, y, z, t)$ can be visualized as a field of ellipsoids (3 orthogonal vectors for every point). Examples include stress, strain, strain-rate.

Notation There are many different notations for scalars, vectors and tensors (and they're often mixed and matched); however, a few of the common ones are

scalars scalars are usually shown in math italics e.g. $f, g, t\ldots$

vectors come in more flavours. when typeset they're usually bold-roman characters e.g. **V**, or the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. When hand written they usually have a line underneath them. Vectors can also be written in component form as $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ or in index notation $\mathbf{v} = v_i\hat{\mathbf{e}}_i$ where $\hat{\mathbf{e}}_i$ is another representation of the unit vectors.

tensors (actually second rank tensors) Typeset in sans-serif font D (or often just a bold $\boldsymbol{\sigma}$), handwritten with two underbars, or by component D_{ij}. 2nd rank tensors are also conveniently represented by matrices.

Definitions of basic operations

vector dot product

$$\mathbf{a}\cdot\mathbf{b} = a_ib_i = |\mathbf{a}||\mathbf{b}|\cos\theta \tag{A.1.1}$$

is a scalar that records the amount of vector **a** that lies in the direction of vector **b** (and vice versa). θ is the smallest angle between the two vectors.

vector cross product $\mathbf{c} = \mathbf{a}\times\mathbf{b}$ is a vector that is perpendicular to the plane spanned by vectors **a** and **b**. The direction that **c** points in is determined by the right hand rule. Note $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$. The cross product is most easily calculated as the determinant of the matrix

$$\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{A.1.2}$$

or

$$\mathbf{c} = (a_yb_z - a_zb_y)\mathbf{i} - (a_xb_z - a_zb_x)\mathbf{j} + (a_xb_y - a_yb_x)\mathbf{k} \tag{A.1.3}$$

or in index notation as $c_i = \epsilon_{ijk}a_jb_k$ where ϵ_{ijk} is the horrid permutation symbol.

tensor vector dot product is a vector formed by matrix multiplication of a tensor and a vector $\mathbf{c} = \mathsf{D}\cdot\mathbf{a}$. In the case of stress, the force acting on a plane with normal vector **n** is simply $\mathbf{f} = \boldsymbol{\sigma}\cdot\mathbf{n}$. Each component of the vector is most easily calculated in index notation with $c_i = D_{ij}a_j$ with summation implied over repeated indices (i.e. $c_1 = D_{11}a_1 + D_{12}a_2 + D_{13}a_3$ and so on for $i = 2, 3$.

A.2 Partial derivatives and vector operators

Definitions Given a scalar function of one variable $f(x)$, its derivative is defined as

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{A.2.1}$$

(and is locally the slope of the function). Given a function of more than one variable, $f(x, y, t)$, the partial derivative *with respect to* x is defined as

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, t) - f(x, y, t)}{\Delta x} \tag{A.2.2}$$

i.e. if we sliced the function with a plane lying along x, the partial derivative would be the slope of the function in the direction of x (See Figure A.1a) likewise y or t.

In space in fact it is convenient to consider all of the spatial partial derivatives together in one handy package, the 'del' operator ($\boldsymbol{\nabla}$) a.k.a. the upside down triangle.. In Cartesian coordinates this operator is defined as

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{A.2.3}$$

In combination with vector and scalar fields, the del operator gives us important information on how these fields vary in space. In particular, there are 3 important combinations

the Gradient the gradient of a *scalar* function $f(\mathbf{x})$

$$\boldsymbol{\nabla} f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} \tag{A.2.4}$$

is a *vector* field where each vector points 'uphill' in the direction of fastest increase of the function (See Figure A.2).

the Divergence the divergence of a *vector* field

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{A.2.5}$$

is a scalar field that describes the strength of local sources and sinks. If $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ the field has no sources or sinks and is said to be 'incompressible'.

the Laplacian the Laplacian of a *scalar* field

$$\nabla^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{A.2.6}$$

is a scalar field that gives the local curvature (See Figure A.3).

the Curl the curl of a vector field

$$\boldsymbol{\nabla} \times \mathbf{F} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\mathbf{i} - (\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z})\mathbf{j} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\mathbf{k} \tag{A.2.7}$$

is a vector field that that describes the local rate of rotation or shear.

Other useful relationships Given the basic definitions, there are several identities and relationships that will be important for the derivation of conservation equations.

Gauss' divergence theorem Gauss's theorem states that the flux out of a closed surface is equal to the sum of the divergence of that flux over the interior of that volume (it is actually closely related to the definition of the Divergence). Mathematically

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \mathbf{F} dV \tag{A.2.8}$$

useful identities the first homework will make you show that

1. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$ (i.e. if a vector field can be written as $\mathbf{V} = \boldsymbol{\nabla} \times \mathbf{g}$ then it is automatically incompressible).
2. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$ (a gradient field is irrotational)
3. $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$
4. $\boldsymbol{\nabla} \times [(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V}] = (\mathbf{V} \cdot \boldsymbol{\nabla})[\boldsymbol{\nabla} \times \mathbf{V}]$

Figures A.1–A.5 show some examples of scalar and vector fields and their derivatives.

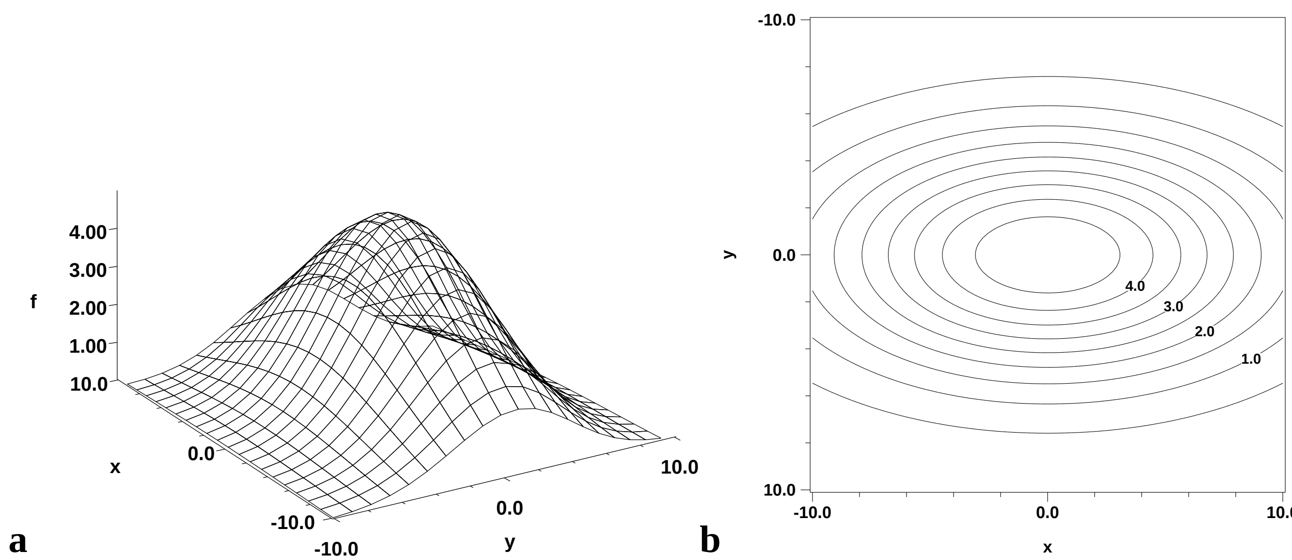

Figure A.1: (a) Surface plot of the 2-D scalar function $f(x, y) = 5 \exp\left[-(x^2/90 + y^2/25)\right]$ (b) contour plot of the same function.

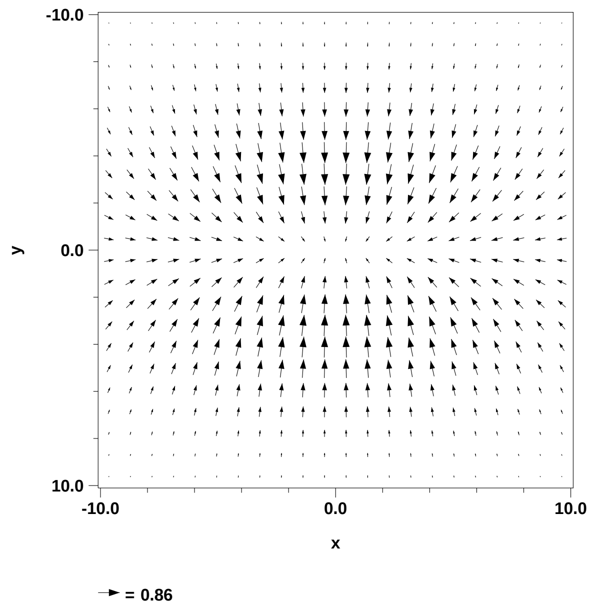

Figure A.2: Vector plot of $\boldsymbol{\nabla} f(x, y)$ for the function f in Figure A.1.

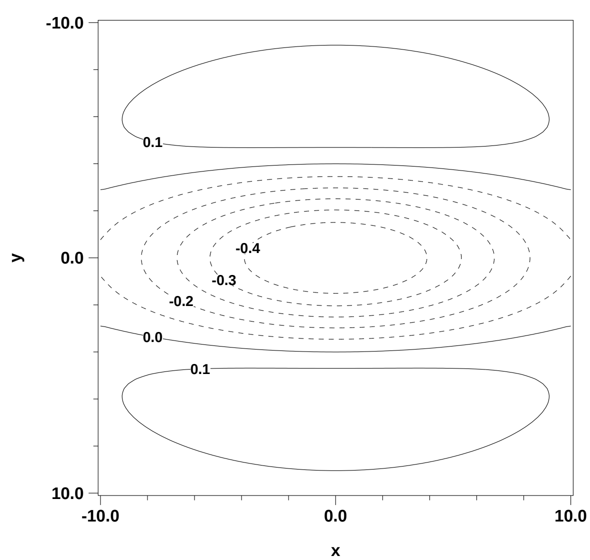

Figure A.3: contour plot of of the divergence of the vector field in Figure A.2. Because $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \nabla^2 f$ this plot is also a measure of the curvature of the function f.

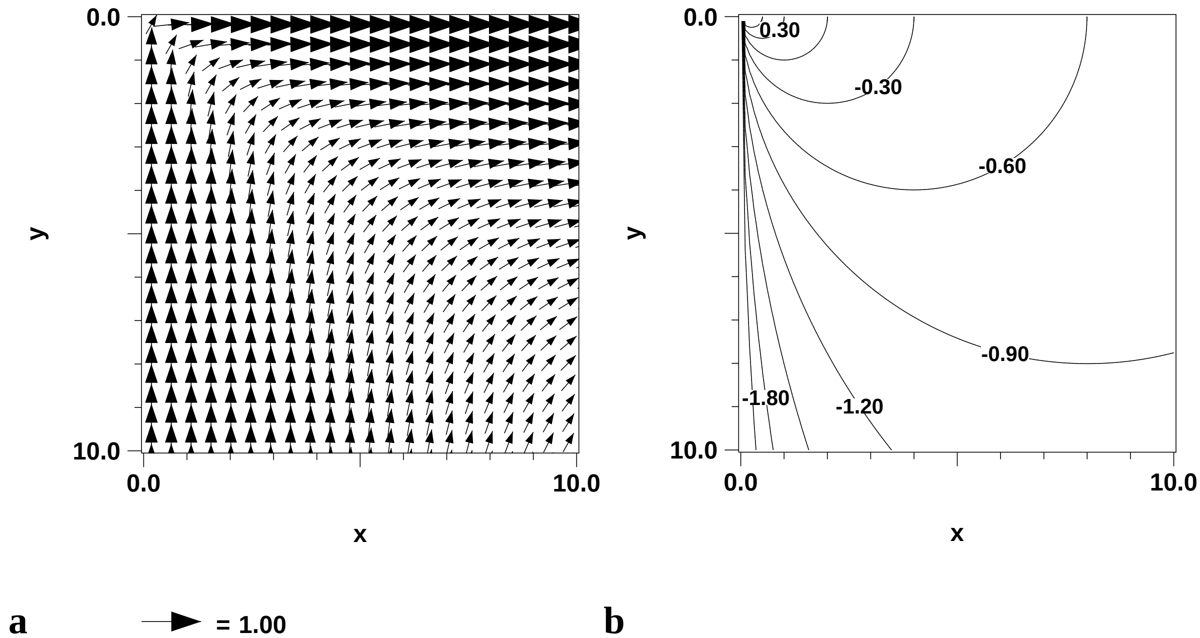

Figure A.4: (a) Vector plot of the 2-D corner flow velocity field $\mathbf{V}(x, y) = 2/\pi\left[\left(\tan^{-1}(x/y) - xy/(x^2+y^2)\right)\mathbf{i} - y^2/(x^2+y^2)\mathbf{j}\right]$ (b) contour plot of $\log_{10}(\boldsymbol{\nabla}\times\mathbf{V}\cdot\mathbf{k})$. The maximum rate of rotation is in the corner. There is no rotation directly on the x axis. This field is incompressible however and $\boldsymbol{\nabla}\cdot\mathbf{V} = 0$

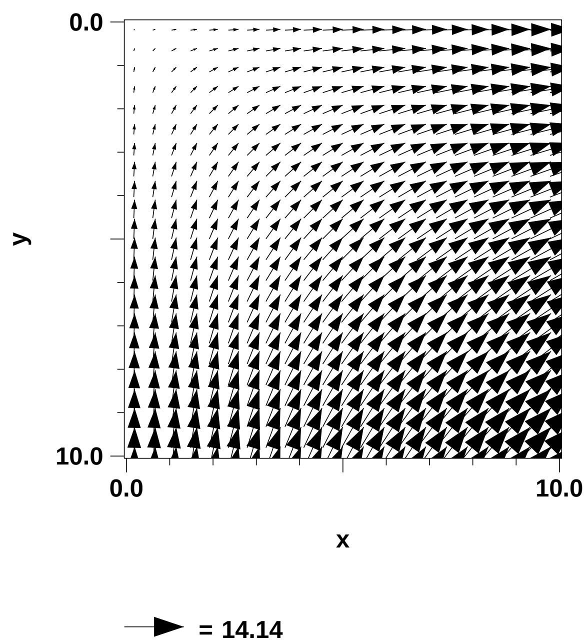

Figure A.5: Vector plot of pure-shear flow field $\mathbf{V}(x, y) = x\mathbf{i} - y\mathbf{j}$. Although the flow lines of this field are superficially similar to those of Figure A.4, this flow is locally irrotational i.e. $\boldsymbol{\nabla}\times\mathbf{V} = 0$. This flow is also incompressible.