1 Problem set #1

Basic Math review, matlab warmup and Conservation equations

Do all problems.

1. Consider the two dimensional function $\psi(x, y) = \sin(m\pi x)\sin(n\pi y)$ on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

   (a) Find $\mathbf{F} = \boldsymbol{\nabla}\psi$

   (b) Find $\mathbf{V} = \boldsymbol{\nabla} \times \psi\mathbf{k}$

   (c) Find $\boldsymbol{\nabla} \cdot \mathbf{F} = \nabla^2\psi$

   (d) Find $\boldsymbol{\nabla} \cdot \mathbf{V}$ and explain why we consider **V** to be "incompressible"

   (e) Use Matlab to create a figure that superposes a vector plot of **V** on a contour plot of ψ for the unit square. Show results for $m = 4$ and $n = 1$. In 2-D, ψ is said to be the "streamfunction" for the velocity field **V**. Why?

2. Eigen fun

   (a) Find *all* solutions to the 2-point boundary value problem

   $$\frac{d^2 f}{dx^2} + \lambda f = 0$$

   on the unit line $0 \leq x \leq 1$ with boundary conditions $f(0) = f(1) = 0$. What are the eigenvalues λ?

   (b) Now consider the discrete eigenvalue problem $A_N\mathbf{x} = \lambda\mathbf{x}$ where A_N is the $N \times N$ tridiagonal matrix

   $$A_N = (N+1)^2 \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$

   Use matlab to find the first 5 eigenvalues of A_N (hint: try help eig) and compare these to the eigenvalues in problem 2a for $N = 5, 10, 20, 100$ (a plot might help). **Extra credit:** How does the difference between the nth continuous and discrete eigenvalue scale with N? Suggest why.

3. The constitutive relationship between stress and strain for a *compressible* viscous fluid can be written in component form as

$$\sigma_{ij} = -P\delta_{ij} + \eta\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) + (\zeta - \frac{2\eta}{3})\frac{\partial V_k}{\partial x_k}\delta_{ij}$$

where P is the pressure, η is the shear viscosity, ζ is the bulk viscosity, $\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ is the fluid velocity, and δ_{ij} is the kronecker delta function ($\delta_{ij} = 1$ if $i = j$, else $\delta_{ij} = 0$). (note: repeated subscripts imply summation, i.e. $\partial V_k/\partial x_k = \boldsymbol{\nabla}\cdot\mathbf{V}$)

   (a) Show that if the viscosities are constant, the divergence of the stress tensor $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \partial\sigma_{ij}/\partial x_j$ can be written in vector notation as

   $$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = -\boldsymbol{\nabla}P - \eta\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{V} + (\zeta + 4\eta/3)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{V})$$

   (b) Show that if the flow is incompressible this reduces to

   $$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = -\boldsymbol{\nabla}P + \eta\nabla^2\mathbf{V}$$

   Here are some identies you might find useful (if you use them... prove them)

   (a) $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$ for any $\mathbf{F}$

   (b) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) = 0$ for any f

   (c) $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{V}) - \nabla^2\mathbf{V}$

4. Derive and scale some conservation equations

   (a) Derive a conservation equation for the concentration of a stable but diffusive trace element in a fluid. The tracer concentration per unit *mass* is c, the fluid velocity is $\mathbf{v}$ and the diffusivity of the tracer in the fluid is D. How does this equation differ from the "heat flow equation"?

   (b) Repeat the exercise for a radioactive trace element whose rate of decay is proportional to the amount of tracer present. Assume the proportionality constant (decay constant) is λ.

   (c) For the problem of a diffusive radioactive tracer, assume that density and diffusivity are constant and derive the *dimensionless* equation for the change in concentration with time ($\partial c/\partial t$), with a scaling such that there is only one adjustable parameter in the problem. What is this parameter and what does it mean physically?