CHAPTER 4

Spherical Coordinates, and Surface Harmonics

So far we have emphasized the use of cartesian coordinates, (x_1, x_2, x_3). But for many practical purposes, especially in the Earth Sciences when we are considering large-scale problems such as the study of global circulation or the propagation of seismic waves to great distances, it is more appropriate to work with spherical polar coordinates (r, θ, ϕ).

Expressions such as ∇V, $\nabla \cdot \mathbf{u}$, $\nabla \times \mathbf{u}$, $\nabla^2\phi$, $\nabla^2\mathbf{u}$, which are independent of any coordinate system, are easily interpreted in cartesians in terms of partial derivatives with respect to x_1, x_2, and x_3. Interpretation of the same expressions in terms of partial differentiation with respect to r, θ, and ϕ is somewhat more difficult, because the directions in which of each of these three coordinates increase, are themselves functions of position. We can label these three directions as the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$, respectively. They correspond to the unit vectors $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ for cartesians (Mark Cane uses $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$). Taking into account the fact that $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ vary in direction as a function of position, unlike the vectors $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$, it can be shown for the scalar $V = V(\mathbf{x})$ that

$$\nabla V = \left(\frac{\partial V}{\partial r}, \frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\right), \tag{4.1}$$

and for the vector $\mathbf{u} = \mathbf{u}(\mathbf{x})$ that

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial\,(r^2\,u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\,(\sin\theta\,u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}. \tag{4.2}$$

Combining these two results, it follows that the Laplacian operator, applied to a scalar, is interpreted in spherical polar coordinates as

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}. \tag{4.3}$$

Surface harmonics are the special functions that are used to describe the lateral variation of properties over the surface of a sphere. Thus, surface harmonics are functions of (θ, ϕ). Symbolized in various different ways, such as $Y_l^m(\theta, \phi)$ or $P_l^m(\cos\theta)e^{\pm im\phi}$, these functions are used, in spherical geometry, in essentially the same way that special functions such

BOX **4.1**

Spherical surface harmonics

A long list of important properties can be derived for the special functions $\Theta(\theta)\Phi(\phi)$ that separate the horizontal variation of solutions to $c^2\nabla^2 P = \partial^2 P/\partial t^2$ in spherical geometry. We here outline the formal derivation of some of these properties, which are needed frequently in geophysics because of the need to define continuous bounded functions over spherical surfaces within the Earth.

Trying a solution $P(\mathbf{x}, t) = R(r)\Theta(\theta)\Phi(\phi)\exp(-i\omega t)$, we find from (2.25) that

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\omega^2 r^2}{c^2}\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}.$$

The left-hand side is independent of ϕ, hence $(1/\Phi)(d^2\Phi/d\phi^2)$ is a constant. Solving for Φ and noting that $\Phi(\phi)$ must be periodic with period 2π if $P(\mathbf{x}, t)$ is to be a single-valued function of position, we find that the eigenfunctions associated with the azimuthal coordinate are

$$\Phi = e^{im\phi} \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{1}$$

The equation in (r, θ) for R and Θ is now

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\omega^2 r^2}{c^2} = \frac{m^2}{\sin^2\theta} - \frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right),$$

where it has been arranged that the left-hand side depends only on r and the right-hand side only on θ. The equation can thus be satisfied for all (r, θ) only if there is some constant K for which

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\frac{\omega^2 r^2}{c^2} - K\right)R = 0 \tag{2}$$

and

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \left(\frac{m^2}{\sin^2\theta} - K\right)\sin\theta\Theta \tag{3}$$

We continue with an analysis of the Θ-equation, beginning with:

THE CASE $m = 0$

The function $\Phi(\phi)$ is constant, and the solution $P(\mathbf{x}, t)$ has axial symmetry. Θ satisfies $d/d\theta(\sin\theta d\Theta/d\theta) = -K\sin\theta\Theta$, and it is convenient to get away from the angle θ and use instead the variable $x = \cos\theta$, since then the trigonometric terms in the Θ-equation are suppressed. We find

BOX **4.1** (continued)

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + K\Theta = 0, \tag{4}$$

known as the *Legendre equation.* For general values of the constant K, the solutions have singularities at the end-points of the range $-1 \le x \le 1$. This is the range corresponding to $0 \le \theta \le \pi$, which is needed to describe position in the Earth. But for certain special values of K, there are nonsingular solutions Θ that turn out to be polynomials in x.

To prove these statements, one assumes a power series solution exists in the form

$$\Theta(x) = b_0 x^k + b_1 x^{k+1} + \ldots = x^k \sum_{i=0}^{\infty} b_i x^i \quad (b_0 \ne 0). \tag{5}$$

The challenge here is to find the value of k (the power of x which starts the series), and all the coefficients. Substituting (5) into (4) and equating the coefficient of each power of x to zero, we find

$$b_0 k(k-1) = 0, \tag{6}$$

$$b_1(k+1)k = 0, \tag{7}$$

and, in general,

$$b_{i+2} = b_i \left[\frac{(k+i)(k+i+1) - K}{(k+i+1)(k+i+2)}\right]. \tag{8}$$

From (8) we see in general that $|b_{i+2}/b_i| \to 1$ as $i \to \infty$. Thus, by comparison with a geometric series, there is convergence of (5) provided $-1 < x < 1$. But what happens at $x = \pm 1$ ($\theta = 0$ or π)? It can be shown, for such x values, that the infinite series for $\Theta(x)$ will *diverge*, unless one of the even-suffix b_i is zero and one of the odd-suffix b_i is zero. (For then all further b_i are zero, so that the infinite series is reduced to a polynomial, which clearly does "converge" for all values including the special values $x = \pm 1$.)

Given that $b_0 \ne 0$, (6) requires $k = 0$ or 1.

Looking at (8) with $k = 0$, we see that the only way to stop the even power series from having an infinite number of terms is if $K = i(i+1)$ for some even integer i. Then $b_i \ne 0$ but $b_{i+2} = b_{i+4} = \ldots = 0$. The only way to stop the odd power series is to require that $b_1 = 0$, which via (8) means that all coefficients of odd powers vanish, and also (7) is satisfied.

Looking at (8) with $k = 1$, we see that the power series for Θ starts with the term $b_0 x$. The only way to stop the odd power series is to require that $K = (i+1)(i+2)$ for some even integer i. If $k = 1$, then (7) requires that $b_1 = 0$, and it follows from (8) that there are no even terms.

We have obtained the important result that the constant K, which was introduced to separate the radial equation from the Θ-equation, must in general be the product of two successive integers. Otherwise, the Θ-equation does not have a solution valid throughout the range $0 \le \theta \le \pi$.

BOX **4.1** (continued)

Furthermore, if $K = l(l+1)$ and l is even, then it is the even powers of x that make up the solution. Similarly, if $K = l(l+1)$ and l is odd, then the solution consists only of odd powers of x. In either case, the solution for Θ is a polynomial of order l, with other terms of order $l-2, l-4, \ldots$, with a lowest-order term of order 1 (if l is odd) or 0 (if l is even). The customary choice for b_0 is made by requiring

$$\Theta(x) = 1 \qquad \text{for } x = 1. \tag{9}$$

The polynomials that result are the *Legendre polynomials.* Writing them out as a sum of descending powers, a great deal of manipulation gives, for l either even or odd, the expression

$$\Theta = P_l(x) = \frac{(2l)!}{2^l(l!)^2}\left[x^l - \frac{l(l-1)x^{l-2}}{2(2l-1)} + \frac{l(l-1)(l-2)(l-3)x^{l-4}}{2\cdot 4\cdot(2l-1)(2l-3)} - \cdots\right], \tag{10}$$

stopping at either x or 1 (times a constant) as the last term. The first few Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2 = \tfrac{1}{2}(3x^2 - 1),$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x), \qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3),$$

and, in general,

$$P_l(x) = \frac{1}{2^l!dx^l}(x^2-1)^l,$$

which is known as *Rodrigues' formula.*

THE CASES $m \neq 0$

We shall initially assume the integer m is positive. Then with $x = \cos\theta$ in (3), we find

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] = \frac{m^2\Theta}{1-x^2} - K\Theta. \tag{11}$$

We might attempt a power-series solution like (5). However, this approach becomes difficult because the formula for b_{i+2} turns out to involve not just b_i (as it did before for Θ with $m = 0$), but also b_{i+1}, and the general solution of such a three-term recursion relation is quite complicated. To guess at an alternative approach, we recall that for $m = 0$ the properties of Θ near $x = \pm 1$ are important. They may also be expected to be important for $m > 0$, by inspection of the coefficients in (11). We thus turn to a brief examination of Θ near $x = \pm 1$. With $\varepsilon = x \pm 1$ and ε small, (11) is approximately

$$\varepsilon\frac{d^2\Theta}{d\varepsilon^2} + \frac{d\Theta}{d\varepsilon} - \frac{m^2\Theta}{4\varepsilon} = 0,$$

which has solutions $\Theta = \varepsilon^{m/2}$ and $\varepsilon^{-m/2}$. The second solution is not well-behaved at $\varepsilon = 0$, and can be rejected. It seems then that Θ should have zeros of order $m/2$ at $x = \pm 1$. They can both be factored out by writing

$$\Theta(x) = (1-x^2)^{m/2}A(x),$$

and we can hope to study Θ by studying $A(x)$.

BOX **4.1** (continued)

This approach turns out to be fruitful, because A satisfies the ordinary differential equation

$$(1-x^2)\frac{d^2A}{dx^2} - 2(m+1)x\frac{dA}{dx} + [K - m(m+1)]A = 0, \tag{12}$$

which *does* have just a two-term recursion formula for the coefficients in an expansion of the form $A(x) = x^k \sum_{i=0}^{\infty} c_i x^i$. The recursion formula turns out to be

$$c_{i+2} = \frac{[(i+m)(i+m+1) - K]}{(i+1)(i+2)} c_i.$$

In general, this formula will generate two series-solutions for $A(x)$ (one of even powers of x, and one of odd powers). If these series were not terminated at some power x^r, they would behave like $(1-x^2)^{-m}$. The requirement that Θ have no singularities in $-1 \le x \le 1$ $(0 \le \theta \le \pi)$ thus leads to the result $c_{r+2} = 0$ for some r. The series ends with the power x^r. It begins with the power x^0 (i.e., a constant) if r is even, and with the power x if r is odd. Thus

$$(r+m)(r+m+1) = K,$$

and K has eigenvalues that again (i.e., as for $m = 0$) are the product of consecutive integers; $r \ge 0$, $m \ge 0$, hence we take $K = l(l+1)$ for some integer $l \ge 0$. Since $r \ge 0$, we find also the important result $m \le l$.

Since K takes the same eigenvalues if $m = 0$ or $m > 0$, the radial function $R(r)$, determined from (2), is unchanged by dropping the requirement of axial symmetry. So the radial functions are independent of m.

We have shown that $\Theta(x) = (1-x^2)^{m/2}A(x)$ where A is now a polynomial in x. There is no difficulty in finding the coefficients of this polynomial. However, a quick way to get an explicit formula for A is available, since, if the equation satisfied by Legendre polynomial P_l (see (4)) is differentiated m times, the result is

$$(1-x^2)\frac{d^{m+2}}{dx^{m+2}}P_l - 2(m+1)x\frac{d^{m+1}}{dx^{m+1}}P_l + [l(l+1) - m(m+1)]\frac{d^m}{dx^m}P_l = 0.$$

Comparing this with the equation (12) satisfied by $A(x)$, we see that a solution for A is $A(x) = d^m P_l(x)/dx^m$. Since $P_l(x)$ is a polynomial involving nonnegative powers of x, there is no danger of $A(x)$ blowing up anywhere in $-1 \le x \le 1$.

The product $(1-x^2)^{m/2} d^m P_l(x)/dx^m$ is therefore a solution for the angular function $\Theta(x)$. It is called the *associated Legendre function*, denoted by $P_l^m(x)$.

The equation (11) for Θ depends upon m only via m^2. Therefore, if $m < 0$, the nonsingular solution must be proportional to $P_l^{|m|}(\cos\theta)$. We adopt the convention

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x), \tag{13}$$

in which the constant of proportionality has been chosen so that

$$P_l^m(x) = \frac{(1-x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{14}$$

applies for all (l, m) such that $-l \le m \le l$.

BOX **4.1** (continued)

Several books have been written on properties of P_l and P_l^m (e.g., Robin 1957; Hobson, 1955), and Wiggins and Saito (1971) and Masters and Richards-Dinger (1998) showed how to compute these functions efficiently. Summarizing the most important formulas, it is known that

$$\frac{1}{(1+r^2-2r\cos\theta)^{1/2}} = \sum_{l=0}^{\infty} r^l P_l(\cos\theta) \quad 0 < r < 1 \tag{15}$$

$$(l-m+1)P_{l+1}^m(x) - (2l+1)xP_l^m(x) + (l+m)P_{l-1}^m(x) = 0 \tag{16}$$

$$(1-x^2)\frac{d}{dx}P_l^m(x) = (l+1)xP_l^m(x) - (l-m+1)P_{l+1}^m(x). \tag{17}$$

It is convenient to define fully normalized surface harmonics

$$Y_l^m(\theta,\phi) = (-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} \tag{18}$$

for integers $l \geq 0$ and integers m such that $-l \leq m \leq l$. Then

$$Y_l^{-m}(\theta,\phi) = (-1)^m[Y_l^m(\theta,\phi)]^*,$$

the * denoting a complex conjugate.

The Legendre functions are orthogonal, as are the azimuthal functions $e^{im\phi}$, and the normalizing factor in (18) has been chosen so that the orthogonality of the $Y_l^m(\theta,\phi)$ takes a simple form, namely

$$\int_0^{2\pi} d\phi \int_0^{\pi} [Y_l^m(\theta,\phi)]^* Y_{l'}^{m'}(\theta,\phi)\ \sin\theta\ d\theta = \delta_{ll'}\delta_{mm'}. \tag{19}$$

(*Note*: this is an integration over the surface of a sphere of unit radius.)

If ψ is the angle between the two directions out from the center of coordinates to the points specified by (θ,ϕ) and (θ',ϕ') in spherical polars, then $\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$, and

$$P_l(\cos\psi) = \frac{4\pi}{2l+1}\sum_{-l\leq m\leq l} Y_l^m(\theta,\phi)[Y_l^m(\theta',\phi')]^*. \tag{20}$$

In the theory for excitation of normal modes by a point source, we need values of Y_l^m and some of its derivatives at $\theta = 0$. A key result is

$$P_l^m(\cos\theta) \to \frac{1}{2^m m!}\frac{(l+m)!}{(l-m)!}\theta^m \qquad \text{as } \theta \to 0, \quad \text{for } m \geq 0.$$

Finally, we comment that surface harmonics provide the horizontal wavefunctions needed to study wave propagation in spherical geometry, and in this sense play roles similar to those of $\cos(k_x x + k_y y)$ and $J_m(kr)e^{im\phi}$ for cartesian and cylindrical geometry.

as $e^{i(k_1x_1+k_2x_2)}$, or $\sin k_1x_1 \sin k_2x_2$, are used in cartesian geometry, where the horizontal variables are (x_1, x_2).

For example, consider the scalar wave equation (2.25), for pressure in a fluid:

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \nabla^2 P. \tag{2.25 again}$$

A detailed derivation of the basic properties of the Legendre polynomials $P_l(\cos\theta)$, the associated Legendre functions $P_l^m(\cos\theta)$, and surface harmonics $P_l^m(\cos\theta)e^{\pm i\phi}$, is given in Box 4.1. These functions provide the separated solutions $\Theta(\theta)\Phi(\phi)$, needed to obtain solutions to (2.25) in the form

$$P(\mathbf{x}, t) = R(r)\Theta(\theta)\Phi(\phi)T(t).$$

Figure 4.1 shows a number of examples of Legendre functions, plotted as large-amplitude topography added to a circle.

A useful way to think of the results derived in Box 4.1, is that the properties of surface harmonics are derived directly from the equations these special functions satisfy, in much the wame way that we could (if we chose) investigate the properties of the special functions $\sin\lambda x$ and $\cos\lambda x$ knowing only that these functions satisfied the equation $\dfrac{d^2 f}{dx^2} + \lambda^2 f = 0$. For example, just from this equation, and a normalization, we could find the power series $\sin\lambda x = \lambda x - \dfrac{\lambda^3 x^3}{3!} + \dots$ and $\cos\lambda x = 1 - \dfrac{\lambda^2 x^2}{2!} + \dots$, and other properties.

Principal results of Box 4.1, in application to (2.25), are that:

(i) The equation for T is $\dfrac{d^2 T}{dt^2} + \omega^2 T = 0$, solved by $T = e^{\pm i\omega t}$.

(ii) The equation for Φ is $\dfrac{d^2\Phi}{d\phi^2} = \text{constant} \times \Phi$, solved by $\Phi = e^{\pm im\phi}$ where m is an integer (since we require that Φ be a single-valued function of position).

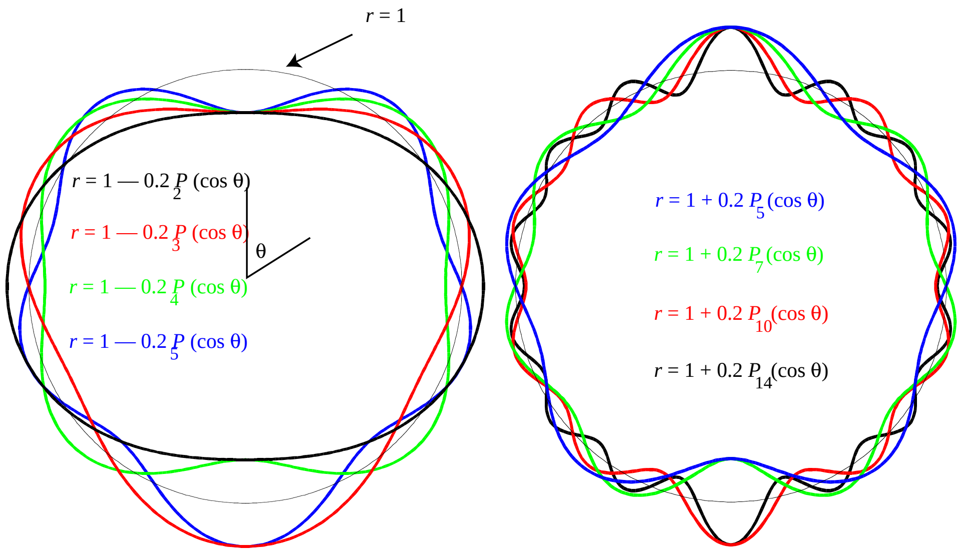
(iii) The equation for Θ is $\dfrac{d}{dr}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right) = \left(\dfrac{m^2}{\sin^2\theta} - K\right)\sin\theta\ \Theta$, which has a well-behaved solution $\Theta(\theta)$ for all θ values in the range $0 \le \theta \le \pi$, but only if K is a product of consecutive integers: $K = l(l+1)$. Note that $-l \le m \le l$. (If K has any value other than $l(l+1)$, the solution for $\Theta(\theta)$ has singularities at $\theta = 0$ and $\theta = \pi$.)

(iv) The equation for R is $\dfrac{d^2(rR)}{dr^2} + \left(\dfrac{\omega^2}{c^2} - \dfrac{l(l+1)}{r^2}\right) rR = 0$. (This result follows from (2) of Box 4.1.) Note that R here depends on l but not on m. We sometimes write this solution as $R_l(r)$ to signify the dependence on l (but not on m).

We can get some perspective on these results, if we consider some general properties of solutions to $\nabla^2 P = 0$ (the Laplace equation) or $\nabla^2 P + \dfrac{\omega^2}{c^2} P = 0$ (the Helmholtz equation), in both cartesians and spherical polars.

For example, the Laplace equation has solutions in cartesians given by

$$P(x_1, x_2, x_3) = e^{\pm ik_1x_1} e^{\pm ik_2x_2} e^{\pm\sqrt{k_1^2+k_2^2}\ x_3}, \tag{4.4}$$



$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = (1/2)(3x^2 - 1) \quad P_3(x) = (1/2)(5x^3 - 3x) \quad P_4(x) = (1/8)(35x^4 - 30x^2 + 3)$

$P_5(x) = (1/8)(63x^5 - 70x^3 + 15x) \qquad P_7(x) = (1/16)(429x^7 - 693x^5 + 315x^3 - 35x)$

$P_{10}(x) = (1/256)(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$

$P_{14}(x) = (1/2048)(5014575x^{14} - 16900975x^{12} + 22309287x^{10} - 14549535x^8$
$+ 4849845x^6 - 765765x^4 + 45045x^2 - 429)$

From these formulas we see that the Legendre polynomials get complicated quickly, as their order increases. But in practice, the actual polynomial expressions are not used for the computation of these shapes. Instead, we can use a so-called recursion relation:

$$(l + 1)\, P_{l+1}(x) = (2l + 1)\, x\, P_l(x) - l\, P_{l-1}(x).$$

From $P_0(x) = 1$ and $P_1(x) = x$ we can quickly compute $P_2(x)$ for a range of x values using this relation. Then we do $P_3(x)$, then $P_4(x)$, etc.

A spreadsheet can easily give the table of values needed to produce Figures such as those above.

FIGURE **4.1**

which has horizontal oscillations and vertical growth or decay (if the horizontal wavenumbers k_1 and k_2 are real). The greater the spatial rate of horizontal oscillation, the greater the rate of exponential growth or decay.

The same Laplace equation solved in spherical polars leads to

$$P(r, \theta, \phi) = \left(ar^l + \frac{b}{r^{l+1}}\right) P_l^m(\cos\theta) e^{\pm im\phi} \tag{4.5}$$

(this is because the radial equation becomes $\dfrac{d^2(rR)}{dr^2} = \dfrac{l(l+1)}{r^2} rR$, solved by $R = r^l$ or $R = r^{-(l+1)}$). But (4.5) is just like (4.4), in having horizontal oscillations (the angular wavenumbers l and m are real integers), and vertical growth (the r^l solution) or decay (like $r^{-(l+1)}$).

Note that when forming the double spatial derivatives that make up $\nabla^2 P$, the horizontal derivatives give a negative result ($-(k_1^2 + k_2^2) \times P$ for cartesians, $-\dfrac{l(l+1)}{r^2} \times P$ for spherical polars), and the vertical derivatives must give the balancing positive result if we wish to solve Laplace's equation, i.e. to ensure that $\nabla^2 P = 0$.

Solutions to the Helmholtz equation are different in that there can be oscillations in all three cartesian directions if the horizontal wavenumber $\sqrt{k_1^2 + k_2^2}$ is less than the total wavenumber $\dfrac{\omega}{c}$. The vertical wavenumber, given by $\sqrt{\dfrac{\omega^2}{c^2} - k_1^2 - k_2^2}$, is then real (homogeneous waves). But if $\sqrt{k_1^2 + k_2^2} > \dfrac{\omega}{c}$, then k_3^2 (the square of the vertical slowness) has to be negative and there is growth or decay in the x_3 direction (inhomogeneous waves), not oscillation. In all cases, the spatial derivatives in the Helmholtz equation, applied to the separated solution in cartesians, give the negative result $-(k_1^2 + k_2^2 + k_3^2) \times P$, matched by the positive result $\dfrac{\omega^2}{c^2} \times P$ because we always require $k_1^2 + k_2^2 + k_3^2 = \dfrac{\omega^2}{c^2}$.

The Helmholtz equation solved by separation of variables in spherical polars uses special functions called spherical Hankel functions for $R = R(r)$. These solutions, written as $R(r) = h^{(1)}_{l+\frac{1}{2}}(\frac{\omega r}{c})$ or $R(r) = h^{(2)}_{l+\frac{1}{2}}(\frac{\omega r}{c})$, also turn out to have oscillatory behavior in the r direction if the horizontal wavenumber is less than the total wavenumber, but now the inequality is expressed as $\dfrac{l+\frac{1}{2}}{r} < \dfrac{\omega}{c}$. If $\dfrac{l+\frac{1}{2}}{r} > \dfrac{\omega}{c}$, the spherical Hankel functions exhibit exponential behavior. [1]

The purpose of the above review, is to bring out some characteristic features of solutions to $\nabla^2 P = 0$ and $\nabla^2 P + \dfrac{\omega^2}{c^2} P = 0$. In particular we can see the way in which the horizontal and vertical derivatives contained in ∇^2 must add either to zero (Laplace) or to a negative result that is cancelled by the $\dfrac{\omega^2}{c^2}$ term (Helmholtz). With the Helmholtz equation, there are options in that the solution can either oscillate or have exponential behavior in the vertical direction, according to the rate of oscillation in the horizontal direction. With the Laplace equation, there is no option: provided there is oscillation in the horizontal direction, growth or decay is required in the vertical direction, not oscillation.

---

1. Spherical Bessel functions, written as $j_{l+\frac{1}{2}}(\frac{\omega r}{c})$, are commonly used also, for solving the Helmholtz equation in spherical polars. Unlike spherical Hankel functions, they have no singularity at the origin, $r = 0$. The relation between the three solutions is $j_{l+\frac{1}{2}}(\frac{\omega r}{c}) = \frac{1}{2}\left[h^{(1)}_{l+\frac{1}{2}}(\frac{\omega r}{c}) + h^{(2)}_{l+\frac{1}{2}}(\frac{\omega r}{c})\right]$, which, in the case that $\frac{l+\frac{1}{2}}{r} < \frac{\omega}{c}$, is like representing a standing wave as a sum of upward and downward traveling waves.

4.1 The utility of Legendre polynomials for representing information on a spherical surface

It turns out that Legendre polynomials have properties of orthogonality that make them ideally suited for representing information on the Earth's surface as a function of latitude.

To see this result, suppose that we have a set of polynomials $P_i(x)$, for $i = 0, 1, 2, \ldots, \infty$ and $P_i(x)$ is a polynomial of order i.

We ask, what the properties of the polynomials $P_i(x)$ that would make the whole set useful for purposes of representing any reasonable function $f(x)$, defined on the interval $-1 \leq x \leq 1$, by a series in the form

$$f(x) = \sum_{i=0}^{\infty} a_i P_i(x) \ ? \tag{4.6}$$

To answer this question, we need to fill in some details. For example, we interpret (4.6) to mean that the longer we make the finite series $S_n(x)$, defined by

$$S_n(x) = \sum_{i=0}^{n} a_i P_i(x), \tag{4.7}$$

the better the fit of $S_n(x)$ to $f(x)$, in the sense that E_n is small, where we define

$$E_n \equiv \int_{-1}^{1} (f - S_n)^2 \, dx. \tag{4.8}$$

Surprisingly, there is almost enough information in the above specification of how we plan to use the polynomials, to actually define what the polynomials must be (that is, to provide the coefficients for every power of x in each of the polynomials $P_i(x)$, for $i = 0, 1, 2, \ldots, \infty$). What then, are these polynomials?

We want the coefficients a_i to be chosen to minimize E_n. Therefore,

$$\frac{\partial E_n}{\partial a_i} = -2 \int_{-1}^{1} (f - S_n) P_i \, dx = 0.$$

So

$$\int_{-1}^{1} f P_i \, dx = \int_{-1}^{1} \left(\sum_{j=0}^{n} a_j P_j \right) P_i \, dx. \tag{4.9}$$

If we go on to the $(n+1)^{\text{th}}$ approximation for $f(x)$, that is, $S_{n+1}(x)$, we add $a_{n+1} P_{n+1}(x)$ to $S_n(x)$ and hence $a_{n+1} \int_{-1}^{1} P_{n+1} P_i \, dx$ to (4.9) above. Provided we decide not to change the earlier coefficients a_i for $i = 0, 1, 2, \ldots, n$, used for $S_n(x)$, when we come to evaluate the additional coefficient a_{n+1} needed for $S_{n+1}(x)$, it follows that

$$\int_{-1}^{1} P_{n+1}(x) P_i(x) \, dx = 0 \quad \text{for} \quad 0 \leq i \leq n, \tag{4.10}$$

which is a statement that the $(n+1)^{\text{th}}$ polynomial is orthogonal to all of the previous n polynomials.

These $n+1$ conditions are enough to determine the coefficients of the $(n+1)^{\text{th}}$ polynomial, up to an overall factor. And the overall factor itself can be determined if we use a normalization, such as $P_i(x) = 1$ for $x = 1$.

To see that there is enough information here to obtain all the polynomial coefficients, let us consider a few examples. Thus, we can take $P_0(x) = b_0$. The requirement that $P_0(1) = 1$ then gives $b_0 = 1$, so $P_0(x) = 1$.

Next, take $P_1(x) = c_0 + c_1 x$. We know $0 = \int_{-1}^{1} P_0 P_1 \, dx = (c_0 x + \frac{1}{2}x^2)|_{-1}^{1} = 2c_0$, and $1 = P_1(1) = c_0 + c_1$. So $c_0 = 0$, $c_1 = 1$, and $P_1(x) = x$.

Then for $P_2(x) = d_0 + d_1 x + d_2 x^2$, we have $0 = \int_{-1}^{1} P_0 P_2 \; dx = 2d_0 + \frac{2}{3}$, $0 = \int_{-1}^{1} P_1 P_2 \, dx = \frac{2}{3} d_1$, and $1 = P_2(1) = d_0 + d_2$, giving $P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$.

We see in these three examples that we have indeed obtained the first three Legendre polynomials. It is possible to obtain a formula for the j-th coefficient of the general polynomial $P_i(x)$, and in this way we find that the polynomials defined by the ability to represent functions $f(x)$ on the interval $-1 \le x \le 1$, are exactly the Legendre polynomials defined by equation (10) of Box 4.1.

The above property of Legendre polynomials depends on the “goodness of fit” criterion expressed by (4.8). But there are many other choices we could make — such as

$$E_n \equiv \int_{-1}^{1} (f - S_n)^2 \, w(x) \, dx. \tag{4.11}$$

for some weight function, $w(x)$.

If, for example, $w(x) = \dfrac{1}{\sqrt{1-x^2}}$, then the ends of the interval are strongly weighted. If we represent a function f by $f(x) = \sum a_i P_i(x)$ and minimize E_n defined in (4.11), the polynomials can be shown to obey the orthogonality rule

$$\int_{-1}^{1} P_i P_j w(x) \, dx = 0 \quad \text{if} \quad i \neq j. \tag{4.12}$$

It is possible to show that this weight function leads to the Fourier sine series based on the orthogonality

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin n\theta \sin m\theta \; d\theta = 0 \quad \text{for integers} \quad n \neq m.$$

The polynomials obeying the weighted orthogonality rule (4.12) turn out to be given by $\sin n\theta = P_n(\sin\theta)$, $\sin m\theta = P_m(\sin\theta)$, and note that if $x = \sin\theta$ then $d\theta = \frac{dx}{\sqrt{1-x^2}}$. [Check this claim. It probably needs modification, since $\sin n\theta$ is a polynomial in $\sin\theta$ only if n is an odd integer.]

Another example of a weight function, would be $w(x)$ so strong that a fit at $x = x_0$ is all that matters. Then the rule

$$\int_{-1}^{1} P_i P_j w(x) \, dx = 0 \quad \text{if} \quad i \neq j,$$

together with $w(x) = \delta(x - x_0)$ leads to the Taylor series

$$f(x) = f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x - x_0)^2}{2!}\, f''(x_0) + \ldots$$

Having seen with these examples that different weight functions lead to different sets of polynomials, with each set enabling us to represent a general function $f(x)$ in a different way, we can ask what is special about the Legendre functions that we found when the weight function is a constant. The physical interpretation here, is that all points in the range $-1 \le x \le 1$ are being treated equally, in the goodness of fit expressed by (4.8). This also means that each element of area on the surface of a sphere is being treated equally, when we seek to express a function $f(\theta)$ (which could also be data on the surface of the sphere) as a sum of Legendre functions. If for example the sphere is the surface of the Earth, with radius $r_\oplus$, then the approximation

$$f(\theta) = \sum_{i=0}^{n} a_i P_i(\cos\theta) = S_n$$

is associated with a criterion that we minimize the squared difference between f and S_n, integrated over the whole surface of the Earth. Thus, we minimize

$$E_n = \int_0^\pi (f - S_n)^2\, 2\pi r_\oplus^2 \sin\theta\, d\theta = 2\pi r_\oplus^2 \int_0^\pi (f - S_n)^2 \sin\theta\, d\theta,$$

leading to the orthogonality result,

$$\int_0^\pi P_i(\cos\theta) P_j(\cos\theta)\ \sin\theta\, d\theta = \int_{-1}^{1} P_i(x) P_j(x)\, dx = 0, \quad \text{if} \quad i \ne j.$$

More generally, for surface harmonics $P_l^m(\cos\theta)e^{im\phi}$ and $P_{l'}^{m'}(\cos\theta)e^{im'\phi}$ we find

$$\int_0^{2\pi} d\phi \int_0^\pi P_l^m(\cos\theta) P_{l'}^{m'}(\cos\theta) e^{i(m-m')\phi}\ \sin\theta\, d\theta = 0$$

unless $l = l'$ and $m = m'$. The surface harmonic $Y_l^m(\theta, \phi)$ is proportional to $P_l^m(\cos\theta)e^{im\phi}$, and normalized so that

$$\int_0^{2\pi} d\phi \int_0^\pi [Y_l^m(\theta, \phi)]^* Y_{l'}^{m'}(\theta, \phi)\ \sin\theta\, d\theta = \delta_{ll'}\delta_{mm'},$$

as noted in equation (19) of Box 4.1.

Just as we can write a 3D vector $\mathbf{A}$ as a sum of its components via

$$\mathbf{A} = \sum_{i=1}^{3} A_i\ \hat{\mathbf{x}}_i$$

(where the basis vectors $\hat{\mathbf{x}}_i$ have unit length and are orthogonal, $\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij}$), and the coefficients are given by

$$A_i = \mathbf{A} \cdot \hat{\mathbf{x}}_i,$$

we can similarly write any function $f = f(\theta, \phi)$ defined on the surface of a sphere as the sum of its components via

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\theta, \phi) \tag{4.13}$$

where the coefficients here are given by

$$f_l^m = \int_0^{2\pi} d\phi \int_0^{\pi} f(\theta, \phi)[Y_l^m(\theta, \phi)]^* \sin\theta \, d\theta.$$

In the Earth Sciences, it is routine to express scalars such gravity anomalies, topography, temperature, and many other variables, as a sum of surface harmonics in the form (4.13). Each harmonic component may then be interpreted according to the underlying physical equation (a diffusion equation, Laplace's equation, a Helmholtz equation) with which that scalar variable may be associated. It is also possible to extend the concept of surface harmonics, to study vector functions defined on a spherical surface. Global circulation models, and models of seismic motion, routinely make use of vector fields that are analysed using vector surface harmonics.

[I expect eventually to add material in this chapter, on the way to represent a delta function on a sphere (as a sum over surface harmonics). Also, to discuss the generating function for Legendre functions, and its relation to the way a point source is represented by a sum of solid harmonics.]

Suggestions for Further Reading

Menke, William, and Dallas Abbott. *Geophysical Theory*, New York: Columbia University Press, 1990 (pp 143–154).

Aki, Keiiti, and Paul G. Richards. *Quantitative Seismology*, Sausalito, California: University Science Books, 2002 (chapter 8).