CHAPTER 6

The Diffusion Equation

The diffusion equation, like the wave equation, provides a way to analyse some important physical processes that require evaluation as a function of space and time. But we shall find that diffusion solutions have properties that in several ways are very different from wave solutions.

Examples of physical variables that can diffuse include temperature, and concentration of pollutants. Certain probabilities also diffuse, such as the probability of finding a drunken person at a particular position at a particular time, if he or she strays from an initial position by taking many steps, each of which has an equal chance of being oriented in any direction. Finding the position of a small particle undergoing Brownian motion is similar to that of finding the drunk — it is best expressed in terms of probabilities, satisfying a diffusion equation.

Here, we present the main ideas using heat as an example.

6.1 Heat Flow by Conduction/Diffusion: an Example of the Diffusion Equation

Let us use the symbols ρ to denote mass density, c to denote the specific heat per unit mass, and T to denote the absolute emperature. Then the heat per unit volume is $\rho c T$.

Heat can spread by convection, radiation, and conduction. We shall consider only the latter mechanism, governed by the basic rule that "heat flux is proportional to temperature gradient," that is,

$$\mathbf{q} = -K \nabla T. \tag{6.1}$$

Here, $\mathbf{q}$ is the rate of flow of heat across unit surface area, so it has the units of heat/(area $\times$ time). Heat flows from hot regions to cold, hence the minus sign in (6.1). The experimental fact that heat flux $\mathbf{q}$ is proportional to temperature gradient, introduces the need for a constant of proportionality. Hence the factor K in (6.1). K is called the thermal conductivity.[1]

1. More generally: $\mathbf{K}$ is a second-order tensor, and $\mathbf{q} = -\mathbf{K} \cdot \nabla T$, or $q_i = -K_{ij} \dfrac{\partial T}{\partial x_j}$. But we shall assume an isotropic conductivity, for which $K_{ij} = K\delta_{ij}$, leading to (6.1).

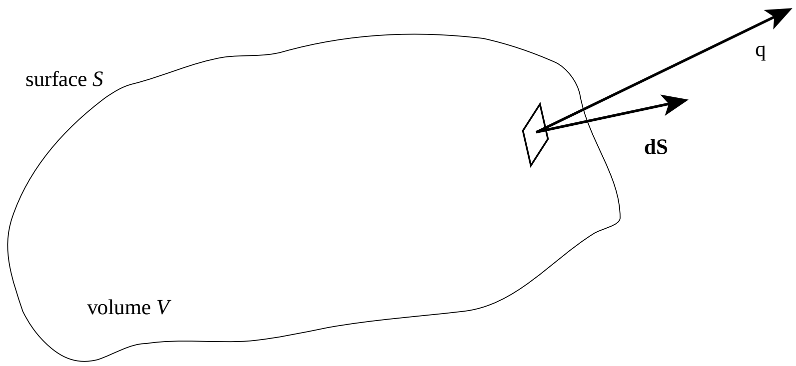


FIGURE **6.1**
An arbitrary volume V is shown, together with its surface S. The flux of heat out of S is characterized by the vector **q**, which need not be orthogonal to the surface. The rate at which heat crosses the area element **dS**, outwards, is given by $\mathbf{q} \cdot \mathbf{dS}$.

Conservation of energy requires that

$$\begin{aligned} &\text{the rate of increase of heat throughout a volume } V \\ &\quad = \text{the rate of production of heat from sources within } V \\ &\qquad - \text{the rate of loss of heat out through the surface } S \text{ of } V. \end{aligned} \tag{6.2}$$

If we denote the heat production per unit volume per unit time by $A(\mathbf{x}, t)$, it follows that

$$\frac{\partial}{\partial t} \int_V (\rho c T)\, dV = \int_V A\, dV - \int_S \mathbf{q} \cdot \mathbf{dS} \tag{6.3}$$

(ignoring convection, radiation, the effects of circulating fluids, and many other phenomena that in practice may lead to additional terms in this equation). V here is any volume in which heat is present, and S is the surface of V (see Figure 6.1).

We can use Gauss's divergence theorem to convert the last term in (6.3) to a volume integral, via

$$\int_S \mathbf{q} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{q}\, dV = -\int \nabla \cdot (K \nabla T)\, dV \qquad \text{from (6.1).}$$

The first term in (6.3) may be written as $\int_V \rho c \dfrac{\partial T}{\partial t}\, dV$, if c is not time dependent, and if we use the fact that ρdV is not time dependent (it is just the constant mass of an original volume element). So all the terms in (6.3) may be collected to give a single volume integral as

$$\int_V \left[\rho c \frac{\partial T}{\partial t} - \nabla \cdot (K \nabla T) - A \right] dV = 0. \tag{6.4}$$

But V is an arbitrary volume. It follows that the integrand in (6.4) must be zero wherever

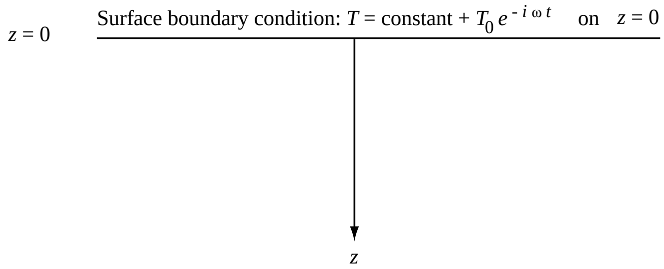


FIGURE **6.2**
A horizontal surface is shown, which is subjected to period heating that maintains temperature as a constant plus a sinusoidal wave with amplitude T_0 and frequency ω (so that the period itself is $2\pi \div \omega$). Taking the depth direction as the z-axis, and assuming no horizontal variation in the temperature, we can use the diffusion equation to find temperature as a function of depth.

it is continuous. [2] Hence

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (K \nabla T) + A, \tag{6.5}$$

which is an example of a diffusion equation. It is a second-order partial differential equation with a double spatial derivative and a single time derivative.

If we assume that K has no spatial variation, and if we introduce the *thermal diffusivity* κ by

$$\kappa = \frac{K}{\rho c}, \tag{6.6}$$

then the diffusion equation assumes a standard form as

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{A}{K}. \tag{6.7}$$

We shall solve this equation in four completely different examples of conductive heat flow, using the solution methods to introduce a number of basic properties of diffusion, and some useful math methods.

6.1.1 PERIODIC HEATING OF THE GROUND SURFACE

Our first example, shows the way in which the constant κ relates the spatial and temporal scales of a simple diffusion problem.

The problem itself is shown in Figure 6.2: a horizontal surface is given a periodic temperature variation, and we wish to find how temperature varies as a function of space and time within the volume of material below the surface.

2. If the integrand were non-zero at a point P, say, we could find a small volume that includes P and that violates (6.4).

In the applied boundary condition on $z = 0$, namely $T = \text{constant} + T_0 e^{-i\omega t}$ (see Figure 6.2), the constant is just the average absolute temperature. There is no source term in the diffusion equation (6.7), so

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \nabla^2 T \tag{6.8}$$

and it is reasonable to seek a solution for $T = T(\mathbf{x}, t)$ in the special form

$$T = T(z, t) = \text{constant} + e^{-i\omega t} Z(z). \tag{6.9}$$

This trial form of solution makes two assumptions: first, that temperature depends spatially only on the depth, and not on horizontal position; and second that the time variation is also of the form $e^{-i\omega t}$, just like the boundary condition. These are reasonable assumptions, given the linearity of the diffusion equation, and the fact that the boundary condition has no horizontal dependence, nor any dependence on initial conditions. (We have simply assumed that the surface boundary condition has always applied, for all times in the past.)

From the boundary condition on $z = 0$, we see from (6.9) that $Z(z) = T_0$ when $z = 0$. But what is the variation of Z with depth? This is where we use the diffusion equation, for substitution of (6.9) into (6.8) gives

$$\frac{-i\omega}{\kappa} Z = \frac{d^2 Z}{dz^2},$$

which is solved by

$$Z(z) = Ae^{-\lambda z} + Be^{\lambda z}, \tag{6.10}$$

where the constant λ is given by $\lambda^2 = \dfrac{-i\omega}{\kappa}$.

The square root of $-i$ is a complex number with amplitude 1 and phase $-\frac{\pi}{4}$ or $\frac{3\pi}{4}$. Note that $e^{-i\pi/4}$ can also be written as $\dfrac{1-i}{\sqrt{2}}$. Without loss of generality, we then have $\lambda = \sqrt{\dfrac{\omega}{\kappa}}\,\dfrac{1-i}{\sqrt{2}}$ and both the solutions $e^{-\lambda z}$ and $e^{\lambda z}$ in (6.10) must be considered.

It is an obvious requirement of the temperature, that it not become infinite as $z \to \infty$. Therefore, $B = 0$ and only the $e^{-\lambda z}$ solution is needed. We noted previously that the boundary condition requires $Z(0) = T_0$, so the constant A is simply T_0 and the complete solution to this problem is

$$T = T(z, t) = \text{constant} + T_0 e^{-\sqrt{\frac{\omega}{2\kappa}}\,z} e^{i\sqrt{\frac{\omega}{2\kappa}}\,z} e^{-i\omega t}. \tag{6.11}$$

The constant here is the same one that appears in the boundary condition. Because of the last exponential factor in (6.11), the temperature penetrates the ground as an oscillation with the same period as the boundary condition. This was our assumption, in the trial form of solution (6.9). But now we have learned from the first exponential factor in (6.11) that the amplitude of this oscillation is damped (attenuated) with depth. At the depth $z_{\frac{1}{2}}$ the amplitude of the temperature fluctuation is a half that at the surface, if

$$e^{-\sqrt{\frac{\omega}{2\kappa}}\,z_{\frac{1}{2}}} = \frac{1}{2}.$$

Since $\log_{10} 0.5 \sim -0.3$ and $\log_e 10 \sim 2.3$, it follows that

$$-\sqrt{\frac{\omega}{2\kappa}}\,z_{\frac{1}{2}} = -0.3 \times 2.3$$

and hence that the “half amplitude depth” is given by

$$z_{\frac{1}{2}} = 0.69 \times \sqrt{\frac{\kappa \times \text{period}}{\pi}} \tag{6.12}$$

where the period of the oscillation is $2\pi/\omega$. Typical values of κ for soil and rock are approximately $0.01\text{cm}^2/\text{s}$. So, the depth at which the temperature oscillation is reduced by a factor of 2 from the surface value, is about 10 cm for a daily oscillation, about 2 meters for an annual oscillation, and about 300 m for a very long period oscillation of 20,000 years (representative of change on the scale of ice ages). Fortunately, when it comes to a decision about how deeply to bury water pipes to avoid freezing in winter, the average temperature at latitudes such as New York is well above freezing, so a depth considerably less than 2 meters is OK. We discuss below the way to combine daily and annual temperature changes, and a general time-dependent boundary condition on $z = 0$.

If we interpret equation (6.12) as a relation between time and distance over which temperature travels, then we see that the distance variable ($z_{\frac{1}{2}}$) is proportional to the square root of the time variable (period), or distance$^2 \propto$ time. This behavior is very different from the way a wave propagates, where for example the dependence of properties is on combinations of space and time such as $(t - x/c)$ or $(t - \mathbf{l} \cdot \mathbf{x}/c)$. The spatial scale and the time scale of a simple wave are proportional, and if we know how long it takes the wave to travel a certain distance, then we expect the wave to travel double the distance in double the time. In constrast, we see here for a simple diffusion solution, that it is the square of the spatial scale which is proportional to the time scale.

From the third exponential factor in (6.11) it follows that the oscillation at depth z is delayed from that at the surface. We can obtain the amount of the delay by writing

$$e^{i\sqrt{\frac{\omega}{2\kappa}}\,z}\,e^{-i\omega t} = e^{-i\omega\left(t - \frac{z}{\sqrt{2\omega\kappa}}\right)}, \tag{6.13}$$

which can be interpreted to say that the change in phase due to depth (the first term on the left-hand side of (6.13)) is equivalent to making a delay in time t by an amount $\dfrac{z}{\sqrt{2\omega\kappa}}$.

Because the diffusion equation is linear, we can directly combine solutions of the type (6.11) to obtain the result of more general boundary conditions. Thus, if

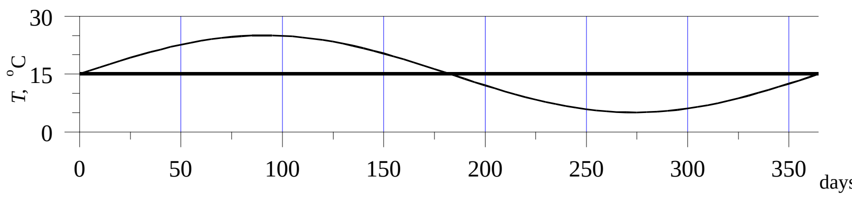


FIGURE **6.3**
An annual variation of the surface temperature is shown — the boundary condition of Figure 6.2, with constant = 15°C, amplitude = 10°C, and $\omega = 2\pi/365$ radians/day.

$$T(z,t)\Big|_{z=0} = \text{constant} + T_1 e^{-i\omega_1 t} + T_2 e^{-i\omega_2 t}$$

for fixed values of T_1, T_2, ω_1, ω_2, then the temperature at depth z is given by

$$T(z,t) = \text{constant} + T_1 e^{-\sqrt{\frac{\omega_1}{2\kappa}}\,z}\, e^{i\sqrt{\frac{\omega_1}{2\kappa}}\,z}\, e^{-i\omega_1 t} + T_2 e^{-\sqrt{\frac{\omega_2}{2\kappa}}\,z}\, e^{i\sqrt{\frac{\omega_2}{2\kappa}}\,z}\, e^{-i\omega_2 t}. \tag{6.14}$$

Figure 6.3 shows the boundary condition for an annual variation, and Figure 6.4 shows a combined annual and daily variation. Each oscillation is attenuated with depth, in the combined solution (6.14).

More generally, the surface boundary condition may be a function of time that we can think of as a summation of its frequency components:

$$\begin{aligned} T(z,t)\Big|_{z=0} &= \text{constant} + f(t) \\ &= \text{constant} + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega) e^{-i\omega t}\, d\omega. \end{aligned} \tag{6.15}$$

Applying the solution (6.11) to each frequency component and superimposing these solutions, we see in this case that T as a function of depth and time is given by

$$T(z,t) = \text{constant} + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega) e^{-\sqrt{\frac{\omega}{2\kappa}}\,z}\, e^{i\sqrt{\frac{\omega}{2\kappa}}\,z}\, e^{-i\omega t}\, d\omega. \tag{6.16}$$

The integrand here requires modification for negative values of frequency, but in practice this solution can be turned into an integral over positive ω values using the results of Problem 3.1. Because of the exponential decay of the integrand in (6.16) as frequency increases, direct numerical integration can easily be made stable.

Typical values of the diffusivity (κ) are: approximately 10^{-6} m^2/s for rocks, and values in the range 10^{-4} to 10^{-5}m^2/s for metals.

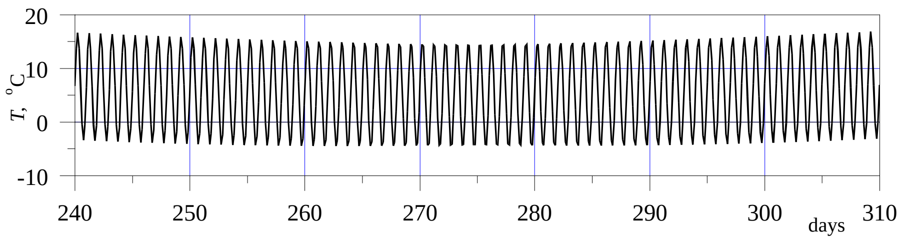


FIGURE **6.4**
For days 240 through 310 of Figure 6.3, an additional daily term is added ($\omega = 2\pi$ radians/day), also with amplitude $= 10°$C. This is a surface boundary condition. The question of how deeply pipes should be buried in winter, to avoid freezing, is crudely answered by finding the value of z in equation (6.14) that filters these inputs ($T_1 e^{-i\omega_1 t}$, $T_2 e^{-i\omega_2 t}$), and keeps T above 0°C.

6.1.2 A SIMPLE MODEL OF CRUSTAL HEAT FLOW

From simple observations made on or near the surface of the Earth, we find that radioactive elements are concentrated in the Earth's crust, and that therefore the materials making up our planet have differentiated and become stratified over time. This conclusion is reached from another solution of the diffusion equation (6.7) which we now discuss, in which we take account of the source term.

A basic observation, is that temperature increases rapidly with depth inside the Earth, as has been noted for hundreds of years by miners. The gradient varies from one location to another, but typical values are on the order of 10° or 15° C per km depth, as measured near the Earth's surface. Presumably this gradient must reduce with depth, otherwise material inside the Earth would melt below about 100 km.

Beginning almost a hundred years ago, measurements of heat flow coming out of the Earth have been made by inserting temperature probes into the ground and carefully measuring the temperature gradient, for example as $\dfrac{\Delta T}{\Delta z} = \dfrac{T(z_1) - T(z_2)}{z_1 - z_2}$. To obtain the heat flow, equation (6.1) can be used. But it is first necessary to know the thermal conductivity K for samples of the same material in which the temperature gradient is obtained in the field. Measurements of K may be done on samples that are brought in to a lab, or sometimes it is possible to measure K directly, in the field.

There are many possible causes of the Earth's observed heat flow. Primordial heat, left over from the time of our planet's formation 4.6 billion years ago, is still coming out as the interior cools. In the vicinity of volcanoes, or at sub-oceanic sites near mid-ocean ridges, hot magma can drive water circulation that vents to the surface or the ocean floor. Various chemical reactions can lead to heating. But a major contributor to the Earth's heat flow is derived from the radioactive decay of isotopes of U (uranium), Th (thorium), and K (potassium).

It has long been observed that there is a simple correlation between measurements of heat flow, and measurements of the heat productivity of rocks found at the Earth's surface. The correlation is indicated in Figure 6.5. In order to interpret observations of the type shown in this Figure, we shall specify a simple model, and use heuristic methods. (That is,

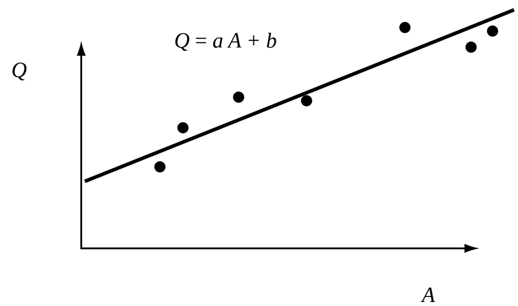


FIGURE **6.5**
A schematic of heat flow measurements and measurements of heat productivity A (see the definition, given just prior to equation (6.3)). The data points are fit approximately by a straight line, which has the form $Q = aA + b$ with two constants, a and b, being the slope and intercept, respectively. We shall develop a simple model, in which each of these constants has a physical interpretation.

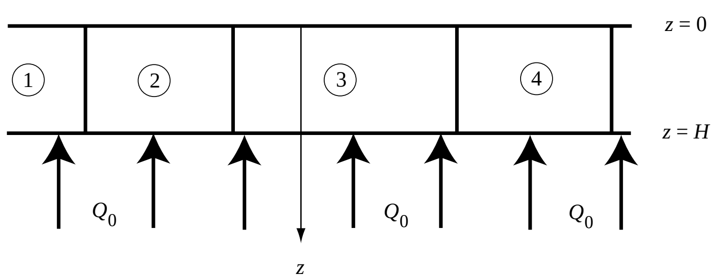


FIGURE **6.6**
A simple model of the Earth's interior is assumed, in which heat producing rocks are confined to the upper layer between $z = 0$ and $z = H$, so that the heat productivity $A = 0$ for $z > H$. Also, it is assumed that the upper layer is composed of separate homogeneous blocks, in each of which the heat productivity A is constant, different for each block (numbered 1, 2, 3, . . .). This productivity can be sampled at the upper surface of each block along with a measurement of the heat flow at the surface (thus giving a data point in Figure 6.5). In each block, heat flow is presumed to be vertical and steady state. Additional heat is due to a deep source which supplies heat flow upwards, across the surface $z = H$, of amount Q_0, which does not change laterally.

we will make simplifying assumptions in order to make progress. The assumptions can be checked later, if necessary.)

Thus, we shall assume the model illustrated in Figure 6.6, in which a series of homogeneous blocks, all with the same thickness b, each has a constant value of heat production A — the source term in (6.7). The constant is in general different in each block, each of which is a crude model of a region specified by its geological and geophysical properties (these properties being different from one block to another). We shall assume that in each block the heat flow is vertical, independent of horizontal position within the block, and also independent of time. Thus our original diffusion equation (6.5) reduces to

$$\frac{d}{dz}\left(K\frac{dT}{dz}\right) = -A$$

in each block. Since we are assuming the heat flow is vertical, we can write the heat flux as $\mathbf{q} = (0, 0, -Q)$. The minus sign is introduced, to acknowledge that we are taking the direction of z increasing as the depth direction, but the heat flow is upwards. Therefore the scalar Q has a positive value, and (6.1) reduces to the scalar equation

$$Q = K\frac{dT}{dz}.$$

These last two equations give

$$\frac{dQ}{dz} = -A$$

for each block, showing that there is a constant heat flow gradient in each block, due to the distribution of heat-producing radioisotopes throughout the volume of the block. We can integrate over the gradient to give the solution

$$Q(z) = -Az + \text{constant}. \tag{6.17}$$

At $z = H$, let us assume there is an upward contribution to the flow of heat as shown in Figure 6.6, of amount $Q_0, comingfrombelow$. The constant in (6.17) is therefore $AH + Q_0$ and the solution for each block is $Q(z) = A(H - z) + Q_0$. It follows that

$$Q(\text{observed}) = Q(z)\Big|_{z=0} = AH + Q_0. \tag{6.18}$$

When this relation is applied to different blocks, it gives the linear relation between observed values of heat flow Q at the surface, and the observed values of heat productivity, shown schematically in Figure 6.5. Furthermore, we see that the intercept b of this Figure is the same as the constant Q_0 of Figure 6.6, and that the slope a of Figure 6.5 equals the layer thickness H of Figure 6.6.

Surprisingly, the slope value a from data plotted as in Figure 6.5 turns out to be only around 5 to 10 km — and definitely less than typical thicknesses of the continental crust. Interpreting this value of a as the thickness of the blocks in our simple model (Figure 6.6), we see that heat productivity has somehow been concentrated into shallow depths within the Earth. This general conclusion is consistent with the gross observation of a high temperature gradient near the Earth's surface, and also with the requirement, noted above, that this temperature gradient must be greatly reduced at lower depths (otherwise the Earth's interior would be largely molten).

The model we have analysed (Figure 6.6) is very simple, and our conclusions may be modified if we make better assumptions. (For example, A could vary with depth in each block in some plausible way, and we need to see how much the heat flow can be affected by topography, and by a component of lateral flow near the vertical boundary between blocks.) But still it appears from observations such as Figure 6.5 that radioactive U, Th, and K, may be concentrated not just in the crust, but in the upper part of the crust — and also that heat flow at the surface has a significant component Q_0 derived from much greater depth.

Q was once measured in "heat flow units" which were microcalories per cm^2 per s. Values in the range around 1 or 2 of these units were common, with higher values (sometimes, much higher) in regions associated with volcanic activity. Today, Q is measured typically in milliwatts per square meter. One classic heat flow unit corresponds to 42 mW/m^2. Pollack et al. (1993)[3] describe and analyse more than 20,000 measurements of Q. They report a global heat loss of 44.2 terawatts, half of which comes from geologically young oceanic lithosphere. The mean heat flows of continents and oceans are 65 and 101 mW m^{-2}, respectively. With the primitive model discussed in this subsection, Q_0 for young oceanic lithosphere is a dominant fraction of the total observed heat flow emerging at the top of oceanic sediments.

The broad interpretation of heat flow measurements can be based upon (6.18).

In the case of measurements made on continental crust, we can think of Q_0 as roughly constant and A as varying laterally for different provinces. The total continental heat flow is derived mostly from heat-producing isotopes that concentrate in the upper crust, though the contribution from below the crust (Q_0) is a significant fraction of the total observed at the surface.

In the case of measurements made on oceanic crust (usually, from measuring dT/dz and K in soft sediments at the ocean floor), again we can use (6.18) but now A is quite small and the dominant contribution is from Q_0, which varies laterally. Pollack et al. (1993)[3] note that measured values of total heat flow for the oceanic crust are observed to vary with crustal age (t, say) according to $Q(t) = Ct^{-1/2}$ for t in millions of years, where the constant C is about 510 mW m$^{-2} = Q(1)$.

6.1.3 DYKE INJECTION

Our third solution of the diffusion equation (6.7) introduces a method of analysis that has wide application to the analysis of linear partial differential equations with a source term.

We suppose that a thin sheet of molten rock has been injected on the plane $z = 0$ at time $t = 0$, and that the amount of heat is C joules per square meter. The heat from this molten rock is conducted away over time, and we seek to obtain the resulting value of temperature T as a function of position z (distance from the source), and time t. We shall assume the sheet of inserted material is in a horizontal plane, and take the z-axis as the depth direction. When the sheet has solidified, and cooled, the inserted rock is referred to as a dyke.

Our first task is to interpret the above description of the source, in terms of a specific form for the heat production term A used in the diffusion equation (6.7). Recall that $A(\mathbf{x}, t)$ is heat production per unit volume per unit time. Therefore

$$A = 0 \quad \text{if} \quad z \neq 0 \quad \text{or if} \quad t \neq 0, \tag{6.19}$$

since the heat in the present problem is introduced only at $z = 0$ and $t = 0$. But although

3. Pollack, H. N., S. J. Hurter, and F. R. Johnson, Heat flow from the Earth's interior: analysis of the global data set, *Reviews of Geophysics*, **31**, 267–280, August 1993.

BOX **6.1**
Dirac delta functions

There are many situations where we would like to idealize values of a function of time or space, so that the function is somehow concentrated at a particular time or place (or both). Typically, we want to express the function in a natural way with a notation that describes when or where the concentration occurs, and what is the overall effect of the concentration. Paul Dirac introduced key ideas on this subject, and we acknowledge his contributions by referring to the idealized functions as Dirac delta functions.

Take, for example, the mass density $\rho(\mathbf{x})$ that describes a point mass m located at position $\mathbf{x} = \boldsymbol{\xi}$. We normally think of ρ as the mass per unit volume defined by the relation

$$\rho(\mathbf{x}) = \lim_{\delta V \to 0} \frac{\delta M}{\delta V}$$

where δM is the mass contained in a volume δV, which is centered on $\mathbf{x}$. It follows that the mass M contained in a volume V is given by

$$M = \int_V \rho(\mathbf{x})\, dV.$$

But for a point mass m located at position $\mathbf{x} = \boldsymbol{\xi}$, the idealized mass density has the following two properties. First, it is zero almost everywhere, so

$$\rho(\mathbf{x}) = 0 \qquad \text{if} \qquad \mathbf{x} \neq \boldsymbol{\xi}. \tag{1}$$

Second, the mass density is strong enough at the point $\mathbf{x} = \boldsymbol{\xi}$ to give a finite result when ρ is integrated over a region of space that includes the point mass itself. Specifically, we have

$$\int_V \rho(\mathbf{x})\, dV = \begin{cases} 0 & \text{if } \boldsymbol{\xi} \text{ is outside } V \\ m & \text{if } \boldsymbol{\xi} \text{ is inside } V. \end{cases} \tag{2}$$

The result here for $\boldsymbol{\xi}$ outside V is an obvious consequence of (1), because the integrand must be zero everywhere. The result with $\boldsymbol{\xi}$ inside V is Dirac's inspiration, which he saw intuitively. The integrand in (2) now has some type of singularity at the point $\mathbf{x} = \boldsymbol{\xi}$, and though the integrand is zero everywhere else, the singularity at this one point is strong enough to give a finite result for the integration.

In practice we don't need to inquire deeply into the nature of the infinity of the integrand at $\mathbf{x} = \boldsymbol{\xi}$, because we can directly get everything we need from the two equations (1) and (2). These equations become the defining property of the Dirac delta function in three-dimensional space. Thus, we introduce the notation $\delta(\mathbf{x} - \boldsymbol{\xi})$ for a function with the two properties

$$\delta(\mathbf{x} - \boldsymbol{\xi}) = 0 \qquad \text{if} \qquad \mathbf{x} \neq \boldsymbol{\xi}, \tag{3}$$

and

$$\int_V \delta(\mathbf{x} - \boldsymbol{\xi})\, dV = \begin{cases} 0 & \text{if } \boldsymbol{\xi} \text{ is outside } V \\ 1 & \text{if } \boldsymbol{\xi} \text{ is inside } V. \end{cases} \tag{4}$$

In terms of this normalized Dirac delta function, the mass density $\rho(\mathbf{x})$ corresponding to a point mass m located at position $\mathbf{x} = \boldsymbol{\xi}$ is given by

$$\rho(\mathbf{x}) = m\delta(\mathbf{x} - \boldsymbol{\xi}). \tag{5}$$

With this notation, we have obtained the desired device for describing where the concentration of mass occurs (it is at $\mathbf{x} = \boldsymbol{\xi}$), and the total amount of the integrated concentration (it is m). If the point mass were at the origin ($\boldsymbol{\xi} = \mathbf{0}$), we would just write $\rho(\mathbf{x}) = m\delta(\mathbf{x})$.

BOX **6.1** (continued)

In one dimension, we can write $\delta(x - \xi)$ for the function of the scalar independent variable x that has the properties

$$\delta(x - \xi) = 0 \quad \text{if} \quad x \neq \xi, \quad \text{and} \quad \int_a^b \delta(x - \xi)\, dx = 1 \quad \text{if} \quad a < \xi < b \tag{6}$$

(that is, if the range of integration includes the value ξ). Using cartesian coordinates, it follows that

$$\delta(\mathbf{x} - \boldsymbol{\xi}) = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2)\delta(x_3 - \xi_3). \tag{7}$$

To obtain an intuitive understanding of the Dirac delta function in the time domain, consider the integrated effect of a force $F(t)$ applied at a point over a period of time from $t = t_1$ to $t = t_2$. The impulse of this force is defined as $P = \int_{t_1}^{t_2} F(t)\, dt$. What then is the impulse of a hammer blow, applied at $t = T$? And what is the corresponding description of the applied force $F(t)$? To obtain a finite impulse P from a hammer blow that is associated with a force F applied only at one instant, we can write $F(t) = P\delta(t - T)$ where the time-domain Dirac delta function is defined by the two properties

$$\delta(t - T) = 0 \quad \text{if} \quad t \neq T, \quad \text{and} \quad \int_{t_1}^{t_2} \delta(t - T)\, dt = 1 \quad \text{if} \quad t_1 < T < t_2. \tag{8}$$

Elsewhere in these notes, we have pointed out additional properties of Dirac delta functions. For example, in Section 3.4 we noted the substitution property, which for continuous functions of time can be written as

$$\int_{t_1}^{t_2} f(t)\delta(t - T)\, dt = f(T) \quad \text{if} \quad t_1 < T < t_2. \tag{9}$$

That Section noted various parallels between Kronecker delta functions and Dirac delta functions, and showed how a formal definition of a Dirac delta function can be obtained as the limit of functions that have a finite range of non-zero values. In Box 2.2 we briefly noted that the Dirac delta function in the time domain, $\delta(t - T)$, can be thought of as the time derivative of the Heaviside step function $H(t - T)$ (which is zero for $t < T$, and unity for $T < t$). Thus,

$$\frac{d}{dt}[H(t - T)] = \delta(t - T).$$

The Fourier transform of $\delta(t - T)$ is easily obtained from an application of (9), and we find

$$\int_{-\infty}^{\infty} \delta(t - T)e^{i\omega t}\, dt = e^{i\omega T}.$$

It follows that the Fourier transform of $\delta(t)$ is just 1. This is an extreme example of the uncertainty principle discussed in Section 3.3.1: if information is totally concentrated in the time domain (like $\delta(t)$), then it must be completely spread out in the frequency domain (equal to a constant value 1, the same for all frequencies).

The final property that we may note for Dirac delta functions, is the practical one that their physical dimension is given by the reciprocal of the independent variable over which these functions are integrated to give a unit result. Thus, from (8), since $\delta(t - T)\, dt$ is dimensionless, it follows that $\delta(t - T)$ has the physical dimension of 1/time, or frequency. Similarly from (6) it follows that $\delta(x - \xi)$ has the physical dimension of 1/length, or wavenumber; and from (4) we see that $\delta(\mathbf{x} - \boldsymbol{\xi})$ or $\delta(\mathbf{x})$ has the dimension of 1/volume. The term $\delta(\mathbf{x} - \boldsymbol{\xi})$ in the right-hand side of (5) therefore contributes not only the concentrated part of ρ (specifying where ρ is not zero), but also the "per unit volume" part of the physical units.

A is zero almost everywhere, as described by (6.19), A is sufficiently strong at $z = 0$ and $t = 0$ that we get a finite amount of heat when integrating A over ranges of z and t that include the place and time for which heat emplacement occurs. That is,

$$\int_{z_1}^{z_2} \int_{t_1}^{t_2} A(z, t)\, dz\, dt = C \tag{6.20}$$

provided $z_1 < 0 < z_2$ and $t_1 < 0 < t_2$.

The two properties given in (6.19) and (6.20) are the defining properties of the Dirac delta function which is concentrated at $z = 0$ and $t = 0$. From the discussion in Box 6.1, we can write

$$A = C\, \delta(z)\, \delta(t). \tag{6.21}$$

Note that the physical dimensions of (6.21) make sense. A is the heat production rate, in units of energy per unit volume per unit time. And on the right-hand side of the equation we see that the units are those of C (energy per unit area), per unit length (from $\delta(z)$), per unit time (from $\delta(t)$). So the units match on both sides of the equation. The interpretation of the right-hand side, is that the delta functions serve to characterize the concentrated (infinite) densities with which energy is pumped into the rock mass, per unit length in the z direction (the spatial density is zero everywhere except at $z = 0$ where it is unbounded), and per unit time (the rate, or temporal density, is zero at all times except at $t = 0$ where it is unbounded). The only detail that we need to know, of the unbounded nature of the source at the special place ($z = 0$) and time ($t = 0$) where it is concentrated, is the integrated effect given in (6.20). Equation (6.21) is simply a restatement, using Dirac delta function notation, of the two earlier equations (6.19) and (6.20).

The relevant diffusion equation, (6.7), has derivatives perpendicular to the z axis that can be ignored in the present problem, since heat is conducted only in the z direction. Since T depends only on z and t, our task is reduced to solving the equation

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} + \frac{C}{K}\,\delta(z)\,\delta(t), \tag{6.22}$$

given that $T =$ constant for all z, and $t < 0$. We shall take this initial constant value as zero. (If it were not zero, we can use the same methods we are about to use, and add the constant back at the end.)

To solve (6.22) we shall take the following three steps:

(i) transform $z \to k$ and $t \to \omega$ by Fourier methods;

(ii) solve algebraically for the transformed solution $T = T(k, \omega)$; and

(iii) invert back to the (z, t) domain by evaluating the inverse Fourier transforms (two of them).

Let us discuss each of these steps in turn.

(i) Equation (6.22) applies for all z and all t (provided we interpret the delta functions as described in Box 6.1). So we can multiply each side of the equation by $e^{-i(kz-\omega t)}$

BOX **6.2**

On Fourier transforms of spatial and temporal derivatives

Working with the Fourier transform pair

$$f(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \quad \text{and} \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t}\, d\omega, \tag{1}$$

we can differentiate the second equation here to obtain

$$\frac{df}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)\; f(\omega)\, e^{-i\omega t}\, d\omega.$$

From examination of the integrand here, it follows that the Fourier time transform of $\frac{df}{dt}$ is $(-i\omega)\; f(\omega)$.

With the spatial transform pair

$$g(k) = \int_{-\infty}^{\infty} g(t)\, e^{-ikz}\, dz \quad \text{and} \quad g(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k))\, e^{ikz}\, dk, \tag{2}$$

the second equation differentiates to give

$$\frac{dg}{dz} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (ik)\, g(k)\, e^{ikz}\, dk.$$

Hence the Fourier space transform of $\frac{dg}{dz}$ is $(+if)\; g(k)$.

Applying these results to a function of both t and z, where

$$f(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,t)\, e^{-i(kz-\omega t)}\, dz\, dt \tag{3a}$$

and

$$f(z,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k,\omega)\, e^{+i(kz-\omega t)}\, dk\, d\omega, \tag{3b}$$

the double Fourier transform of $\frac{\partial}{\partial t} f(z,t)$ is $(-i\omega)\; f(k,\omega)$; and the double Fourier transform of $\frac{\partial^2}{\partial z^2} f(z,t)$ is $(ik)^2\; f(k,\omega) = -k^2\; f(k,\omega)$. These are the results we need to interpret the differentiated terms in equation (6.22) in the (k,ω) domain.

Note that different sign conventions for the Fourier transforms (1) and (2) may give slightly different results (for example, multiplication by $+i\omega$ instead of $-i\omega$). Throughout these notes we are using the mixed convention given in (1) and (2) and discussed in Box 3.1.

and integrate over all real values of z and t. The reason to do this, is that partial differentiation (with respect to t and z) becomes scalar multiplication (by $-i\omega t$ and $+ikz$) in the transform domain, as explained in Box 6.2, leading to an equation for $T = T(k, \omega)$ that is easy to solve as a specific algebraic expression. Specifically, the three terms in (6.22) transform as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{\kappa}\frac{\partial T}{\partial t}\right) e^{-i(kz-\omega t)}\,dz\,dt = \frac{(-i\omega)}{\kappa}T(k,\omega);$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 T}{\partial z^2}\right) e^{-i(kz-\omega t)}\,dz\,dt = (-k^2)T(k,\omega);$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{C}{K}\,\delta(z)\,\delta(t)\right) e^{-i(kz-\omega t)}\,dz\,dt = \frac{C}{K}.$$

This last result is obtained from the "substitution" property of the Dirac delta function, illustrated earlier by (3.22) and equation (9) of Box 6.1. More fundamentally, it can be obtained directly from (6.19) and (6.20) applied to $\frac{A}{K}$ in (6.7).

(ii) Using the three results listed above, equation (6.22) has now been reduced to

$$\frac{-i\omega}{\kappa}T(k,\omega) = -k^2 T(k,\omega) + \frac{C}{K}. \tag{6.23}$$

This is an algebraic equation for the double transform of T, easily solved to obtain

$$T(k,\omega) = \frac{i\kappa C}{K}\frac{1}{\omega + i\kappa k^2}. \tag{6.24}$$

(iii) In order to obtain $T(z,t)$ from $T(k,\omega)$ we must evaluate the two inverse transform integrals. We shall go first from ω to t, then from k to z. Thus

$$T(k,t) = \frac{1}{2\pi}\frac{i\kappa C}{K}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\omega + i\kappa k^2}\,d\omega. \tag{6.25}$$

The integrand in (6.25) has a simple pole at $\omega = -i\kappa k^2$. The quickest way to evaluate this integral explicitly, is to use the methods given in Chapter 5. Thus, although the integration in (6.25) is over purely real values of ω, we shall allow ω to become a complex variable and use the result that

$$\oint_C \frac{f(\omega)}{\omega - \omega_0}\,d\omega = \begin{cases} 0 & \text{if } \omega_0 \text{ is outside the circuit } C\text{, and} \\ 2\pi i\ f(\omega_0) & \text{if } \omega_0 \text{ lies inside the circuit.} \end{cases} \tag{6.26}$$

[This is a basic result in the calculus of residues. C is a closed circuit in the complex ω-plane, taken in the counterclockwise direction. See (5.17).]

First, assume $t < 0$. To the real axis path of integration in (6.25), we add the upper semicircle as shown on the left in Figure 6.7. (The semicircle has arbitrarily large radius). Since $e^{-i\omega t} = e^{\Im[\omega]t}$, the integrand of (6.25) is exponentially small in

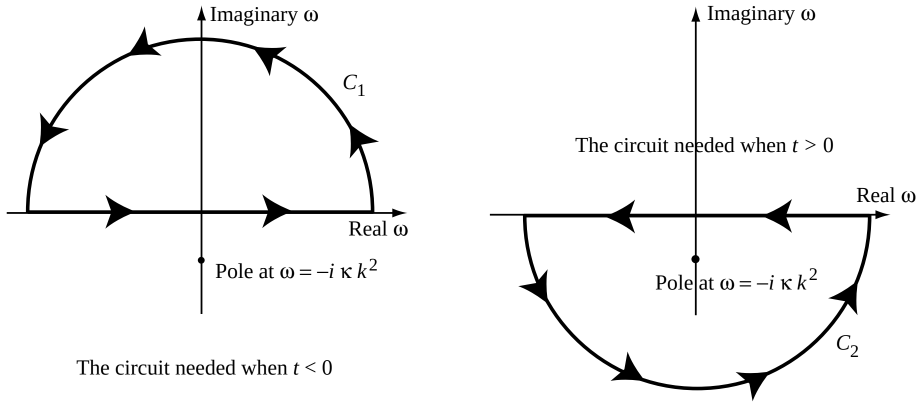


FIGURE **6.7**
Closed circuits in the complex ω-plane, used for evaluating (6.25). For $t < 0$ we use the path C_1 shown on the left, on which $e^{-i\omega t}$ is exponentially small. For $t > 0$ we use the path C_2 shown on the right. Note that we adopt the convention of taking circuit integrals in the counterclockwise direction.

the upper half of the complex ω plane, where $\Im[\omega] > 0$, if $t < 0$. It follows for a large enough semicircle that

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + i\kappa k^2}\, d\omega = \oint_{C_1} \frac{e^{-i\omega t}}{\omega + i\kappa k^2}\, d\omega = 0$$

because the pole lies outside the circuit. Hence $T(k, t) = 0$ for $t < 0$ — confirming a result we expect.

Second, assume $t > 0$. To complete the circuit, we now must add a return path in the *lower* half plane (because in this case, with $t > 0$, $e^{-i\omega t}$ is exponentially small if $\Im[\omega] < 0$). See the right-hand side of Figure 6.7. So, from (6.26)

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + i\kappa k^2}\, d\omega = -\oint_{C_2} \frac{e^{-i\omega t}}{\omega + i\kappa k^2}\, d\omega = -2\pi i\, e^{-\kappa k^2 t}$$

(minus signs arise because the circuit C_2 is by convention taken counterclockwise, and we must reverse this direction to get the integral from negative to positive values along the real ω axis).

At this point we have an explicit solution for the first inverse transform, namely

$$T(k, t) = \frac{C\kappa}{K}\, e^{-\kappa k^2 t}\, H(t) \tag{6.27}$$

where $H(t)$ is the Heaviside function, sometimes called the unit step function ($H(t) = 0$ for $t < 0$, and $= 1$ for $t > 0$).

Finally, we must invert from k to z. We consider only $t > 0$, and our integral is

$$T(z, t) = \frac{C\kappa}{2\pi K} \int_{-\infty}^{\infty} e^{-\kappa t k^2}\, e^{ikz}\, dk. \tag{6.28}$$

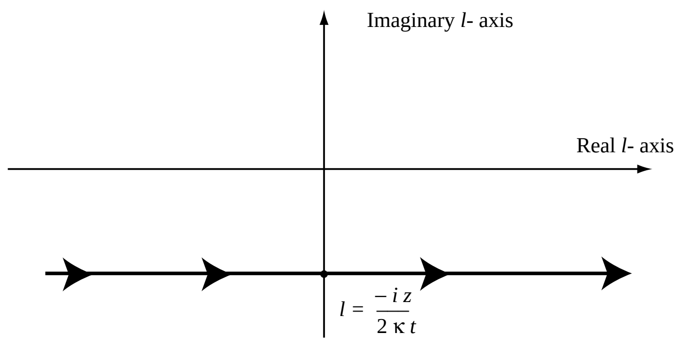
BOX **6.3**

Equivalence of two paths of integration

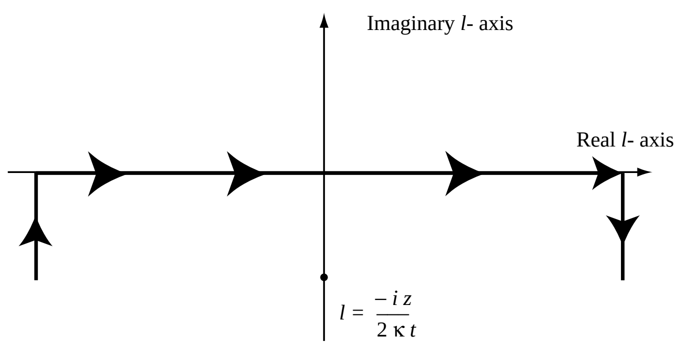
To verify the result given in (6.29), we need to show that

$$\int_{-\infty-\frac{iz}{2\kappa t}}^{\infty-\frac{iz}{2\kappa t}} e^{-\kappa t l^2}\, dl = \int_{-\infty}^{\infty} e^{-\kappa t l^2}\, dl. \tag{1}$$

The path of integration of the left-hand side of (1) is *not* the real l-axis, but a line below it. (The line extends all the way from infinity on the left, through the indicated point in the complex l-plane given by $l = \dfrac{-iz}{2\kappa t}$, then all the way to infinity on the right.)



But instead of the path of integration shown above, we can take the following path:



Since the integrand $e^{-\kappa t l^2}$ is analytic in the region between the paths shown in these two Figures, we can apply (5.7) and equate the result of integrating along the path shown in the upper Figure, to the result using the path shown in the lower Figure. But along the vertical sections, from $l = -\infty - \dfrac{-iz}{2\kappa t}$ to $l = -\infty$ on the real l-axis, and then from the real l-axis again at $l = \infty$ to $l = \infty - \dfrac{-iz}{2\kappa t}$, there is neglible contribution because the integrand is arbitrarily small. Throwing away these two segments, it follows that (1) is valid.

This result, like many applications of the properties of complex functions of a complex variable including our discussion from (6.25) to (6.30), takes a whole lot longer to demonstrate the first time you see it, than to use in practice.

We can force this integrand to look like something we can integrate if we make the substitution

$$-\kappa t k^2 + ikz = -\kappa t l^2 - \frac{z^2}{4\kappa t} \quad \text{where} \quad l = k - \frac{iz}{2\kappa t},$$

for then the term involving $\frac{z^2}{4\kappa t}$ is a constant with respect to the integration, and we have something like

$$\int_{-\infty}^{\infty} e^{-\kappa t k^2 + ikz}\, dk = e^{\frac{-z^2}{4\kappa t}} \int e^{-\kappa t l^2}\, dl = e^{\frac{-z^2}{4\kappa t}} \sqrt{\frac{\pi}{\kappa t}} \tag{6.29}$$

(the last equality here uses a result given in Box 3.2). The only problem with (6.29) is with the limits of integration. For the first equality of (6.29) to be true, the limits of the integral over l are from $-\infty - \frac{iz}{2\kappa t}$ to $+\infty - \frac{iz}{2\kappa t}$. For the second equality of (6.29) to be true, the limits of the integral over l need to be from $-\infty$ to $+\infty$. Does it matter, that these limits of integration are different?

The answer is no, for reasons detailed in Box 6.3.

Putting the results (6.27)–(6.29) together, we obtain at last the result we seek in the space-time domain:

$$T(z, t) = \frac{C\kappa}{2K} \frac{e^{-\frac{z^2}{4\kappa t}}}{\sqrt{\pi \kappa t}} H(t). \tag{6.30}$$

Having executed all of the steps (i), (ii), and (iii), we next comment briefly on two properties of the solution, (6.30).

First, we should expect that heat is conserved in vertical columns for all times $t > 0$. That is, we expect

$$\int_{-\infty}^{\infty} \rho c T(z, t)\, dz = C. \tag{6.31}$$

Indeed the solution (6.30) has this property, for

$$\begin{aligned}
\text{Left-hand side of (6.31)} &= \frac{\rho c\, C\kappa}{2K\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4\kappa t}}\, dz = \frac{\rho c\, C\kappa}{2K\sqrt{\pi\kappa t}} \sqrt{4\kappa t\, \pi} \\
&= C \quad (\text{from Box 3.2 and } K = \rho c\kappa) \\
&= \text{Right-hand side of (6.31).}
\end{aligned}$$

Second, in Figure 6.8 we show the main space-time properties of the solution. Note that at any fixed value of z, the temperature begins to rise instantly as soon as t becomes positive. In this sense, the propagation speed for conduction is infinite. However, in practical terms we don't see this effect because the temperature remains infinitesimal until t has become large enough for $-\frac{z^2}{4\kappa t}$ to become small, thus avoiding exponentially small values of T.

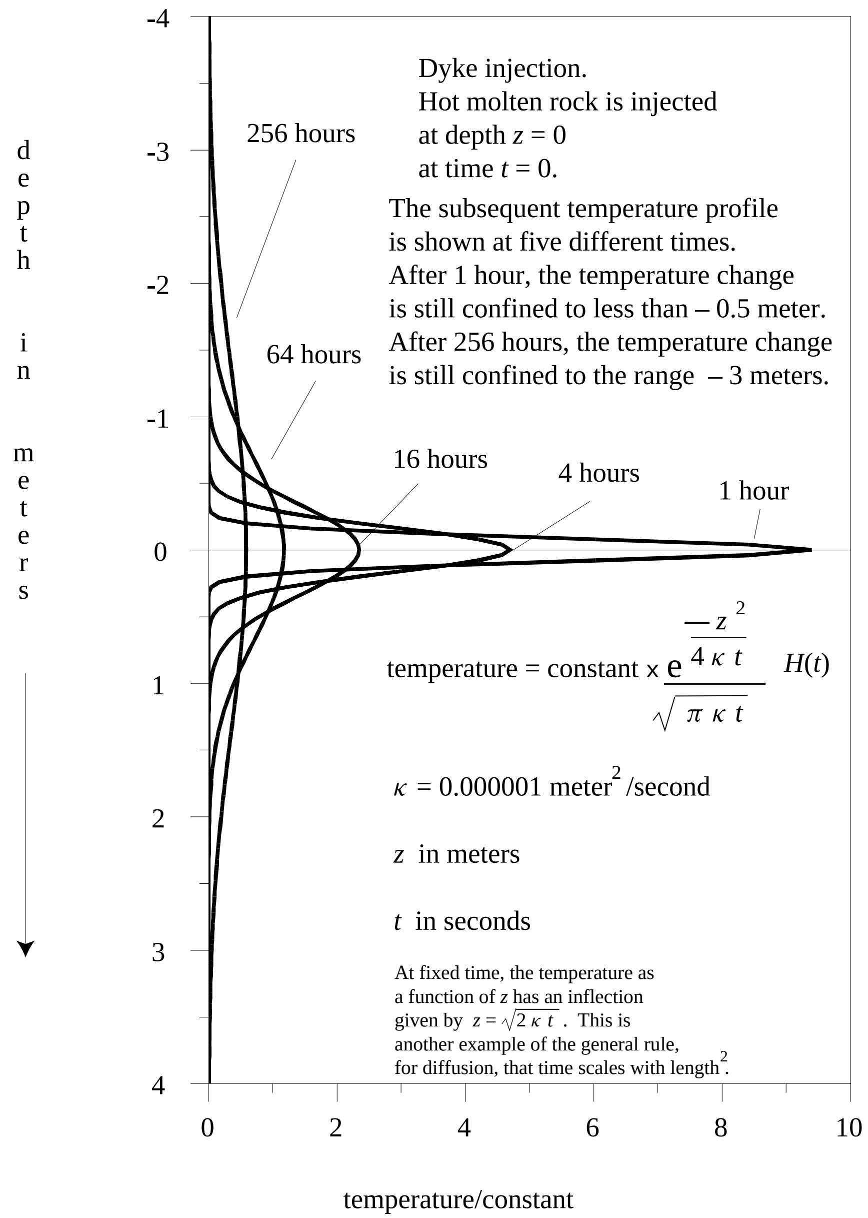


FIGURE **6.8**
Values are shown, of the temperature solution (6.30) for the problem of dyke injection. The constant referred to in this Figure is $C\kappa/(2K)$.

At any fixed time t, the temperature has a gaussian distribution as a function of depth z. The width of the gaussian increases for increasing values of t, and the height of the gaussian correspondingly decreases, thus preserving the total heat (the area under each gaussian), as indicated by (6.31). As noted in Figure 6.8 and repeatedly throughout this chapter, time in the diffusion solution scales in proportion to the square of the distance.

6.1.4 RADIATION TEMPERATURE AT THE SURFACE OF A SPHERICAL CONDUCTING SOLID WITH INTERNAL HEATING

Our fourth and final discussion of a particular solution to the 3D diffusion equation, (6.7), concerns the temperature distribution throughout a homogeneous spherical solid with a constant internal source of heat. The discussion below is based on Menke and Abbott (1990), and is a very simple model of a planet or planetesimal in space.

Thus, we consider a spherical planetary object with radius, say, 6371 km (equal to $r_\oplus$, the radius of the Earth). But we are using only a very crude model, which everywhere has the same conductivity (K), mass density (ρ), specific heat (c), and strength of internal heating (A). Furthermore, we assume the temperature has stabilized, is spherically symmetric, and is no longer time dependent. In this case, the diffusion equation

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{A}{K} \qquad \text{((6.7) again)}$$

reduces to

$$K\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -A \tag{6.32}$$

because the general dependence of T on space and time is replaced by $T = T(r)$. (We have interpreted the spatial derivatives in the Laplacian as given for spherical polars in (4.3), in the especially simple case of dependence only on the radial coordinate, r.)

Equation (6.32) is easily integrated, first to give

$$Kr^2\frac{dT}{dr} = -\frac{1}{3}Ar^3 + B$$

for some constant B. After dividing through by r^2 we can integrate a second time, obtaining

$$KT = -\frac{1}{6}Ar^2 - \frac{B}{r} + C$$

where C is a second integration constant.

We require $B = 0$ since there nothing special about the temperature at the center of the sphere which would make T singular there. And we can express C in terms of the temperature at $r = r_\oplus$, finding that

$$T = \frac{1}{6K}A(r_\oplus^2 - r^2) + T(r_\oplus). \tag{6.33}$$

It is known that the radiation into outer space is proportional to the fourth power of $T(r_\oplus)$ (expressed in degrees Kelvin, °K). The constant of proportionality for this relationship is called the Stefan-Boltzmann constant, σ $(= 5.7 \times 10^{-8}\ \mathrm{W\ m^{-2}\ {}^\circ K^{-4}})$. The heat flux, $\mathbf{q}$, which in spherical polars has components $(q(r), 0, 0)$ with scalar q, is given in terms of the vertical temperature gradient by $q = -K\dfrac{dT}{dr}$. So from (6.33) we find that

$$q\Big|_{r=r_\oplus} = \frac{A}{3} r_\oplus = \sigma T(r_\oplus)^4.$$

The last equality here allows us to obtain the surface temperature in terms of the strength of the internal heat source, A, and the planetary radius. The result

$$T(r_\oplus) = \left(\frac{A r_\oplus}{3\sigma}\right)^{\frac{1}{4}}. \tag{6.34}$$

Even if we use a high value for A (say, 10 microwatts per cubic meter), and assume that it applies throughout the Earth), (6.34) gives a temperature of only about 140 degrees above absolute zero. This basically is why planetary surfaces are so cold — as long as they are not exposed to solar radiation and have no atmosphere. Our own temperate planet would be in bad shape if life had to rely upon internal sources of heat.

Suggestions for Further Reading

Menke, William, and Dallas Abbott. *Geophysical Theory*, New York: Columbia University Press, 1990 (pp 204–217).

Carslaw, Horatio S., and J. C. Jaeger. *Conduction of Heat in Solids*, Oxford University Press. Second edition reprinted, 1973. Second edition in paperback, 1986.

Problems

6.1 Prove the equality (7) of Box 6.1. That is, show that the right-hand side of this equation has the defining qualities of the 3D Dirac delta function.

6.2 Given that the total heat flow out of the Earth's interior is about 44.2 terawatts, what is the average heat flux through the surface of the Earth? Show that this value has the expected properties of (1) lying between the average continental value (65 mW $\mathrm{m^{-2}}$) and the average oceanic value (101 mW $\mathrm{m^{-2}}$), and (2) being somewhat closer to the average oceanic value (oceans have greater surface area than continents).

6.3 Work through some relevant values of A and the value of σ (Stefan-Boltzmann) to see what the surface temperature given by (6.34) actually is.