

Department of Applied Physics and Applied Mathematics
Columbia University
APPH E4210. Geophysical Fluid Dynamics
Spring 2008

Problem Set 5

(Due Mar 27, 2008)

1. Consider the statement for conservation of potential vorticity (PV) for a shallow layer of fluid (see figure):

$$\frac{D}{Dt} \left(\frac{\xi + f}{H} \right) = 0.$$

- (a) Show that in the limit of small Rossby number, this statement is equivalent to the quasigeostrophic potential vorticity equation (QGPVE):

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0,$$

where, the potential vorticity q is given by

$$q = \nabla^2 \psi - \frac{1}{\lambda_d^2} \psi + \beta y + f_o \frac{h_B}{H_o},$$

and, $\psi = g\eta/f_o$ is the streamfunction.

- (b) Show that a single plane Rossby wave is an exact solution of the nonlinear QGPVE. What about a *sum* of plane waves?

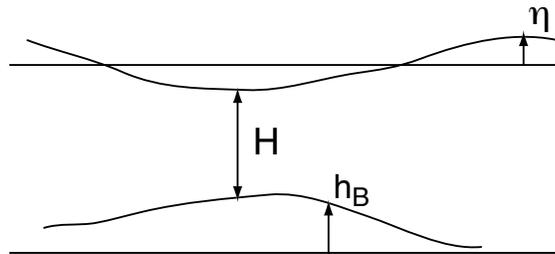


Figure 1: Definition sketch for problem 1.

2. (*Steady internal waves in a background flow.*) Consider an unbounded, stratified, incompressible fluid flowing over sinusoidally varying bottom topography. For simplicity, ignore the effect of rotation, and further assume that the flow is two dimensional (in the x - z plane).

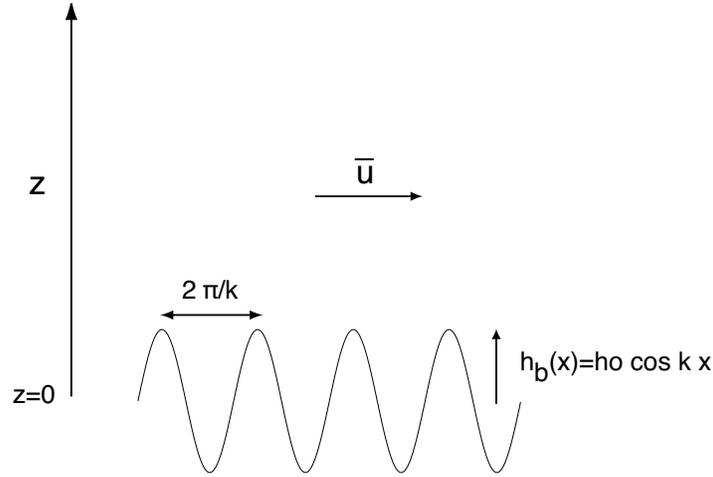


Figure 2: Definition sketch for problem 2.

- Write down the governing equations (momentum, continuity, density) for small perturbations about a steady zonal flow $\bar{u}(z)$. (That is, linearize the governing equations about a basic state flow $(\bar{u}(z), 0, 0)$.)
- Derive a single equation for the relative vorticity of the disturbance. It is convenient to write this equation in terms of a streamfunction ψ for the disturbance field.
- Assuming that the disturbance is steady in time and periodic in x (or alternatively, decays as $x \rightarrow \pm\infty$), show that the vorticity equation is

$$\nabla^2 \psi + \left(\frac{N^2}{\bar{u}^2} - \frac{\bar{u}_{zz}}{\bar{u}} \right) \psi = 0.$$

- For topography of the form $h_b(x) = h_o \cos kx$, what is the linearized boundary condition at $z = 0$ (see sketch).
- Assuming constant N and \bar{u} , and $k^2 > N^2/\bar{u}^2$ (with $k > 0$), find the solution to the equation in part (c). Make a sketch of this solution, showing, in particular, the phase relation of the pressure field to the horizontal topography. (From the various pieces you should be able to relate p' to ψ .) Calculate the vertical flux of energy $\overline{p'w'}$ due to the wave. ($\overline{(\)}$ represents a horizontal average over a wavelength.) Also compute the force exerted by the fluid on the topography. This is the horizontal force (per unit area) exerted by the pressure field on the topography, and (in linearized form) is given by:

$$F = \frac{k}{2\pi} \int_0^{2\pi/k} p'(z=0) (dh/dx) dx$$

Give a physical explanation of this result. (Note that the topography exerts an equal and opposite force on the fluid. This force is known as the *drag*.)

- Repeat part (e), but now for the case $k^2 < N^2/\bar{u}^2$ ($k > 0$). Note that the general solution consists of two parts. To pick the correct solution, you must impose the additional

constraint that the vertical energy flux be upward. This is called the *radiation condition*. Make a sketch of the solution, showing the pressure field and the phase lines of the wave. In what direction do the phase lines tilt with height? Show that the vertical energy flux is equal to the rate at which work is done by the fluid on the topography. Give a physical explanation of this result.

Note: $\bar{u}k$ is the frequency of encounter of the fluid particles with the crests of the topography. As the fluid travels over the bumps, internal gravity waves with this *intrinsic* frequency (i.e., frequency measured in a frame of reference moving with the background wind) are generated. (In a fixed reference frame the frequency is zero, since we are considering stationary waves.) When this frequency is greater than N (part e), the fluid cannot sustain internal gravity waves and the disturbance is *trapped* near the bottom. When it is less than N (part f), the fluid is able to sustain waves and energy can propagate vertically. These waves are known as *lee waves*.