Wavelet Analysis

Wavelet analysis has been applied to many geophysical time series over the past ten years by various authors. In its particulars, it remains open to interpretation and debate. *Hubbard* (1996) gives an excellent introduction to wavelet analysis while *Torrence and Compo* (1998) address many of the practical details of applying wavelet analysis to time series. Here we describe the methods used in this study as well as present some of the limitations of the wavelet transform that were encountered.

The wavelet transform is a useful alternative to the fast Fourier transform (FFT) for spectral analysis. For non-periodic, non-stationary time series, the FFT transform can give spurious results. A wavelet transform, in the most general sense, is the convolution of a given waveform with a function. The waveform (or wavelet) is restricted in that it must integrate to zero over the interval from negative infinity to infinity, as well as over an interval on the order of or shorter than the length of the function being transformed. The convolution is performed for a set of scaled versions of the wavelet. A strong output is given where the shape of the time series is closely matched by the shape of the chosen wavelet. For wavelets which are essentially periodic over their nonzero interval, this response is related to the frequency content of the time series.

An example of such a periodic wavelet is the Mexican hat wavelet. It is the negative of the second derivative of a unit area Gaussian (figure 2), given by:

\[ g(t') = \left( \frac{1}{2\pi} \right)^{1/2} (1 - t'^2) e^{-t'^2/2} \]  

(4)
The wavelet transform $W(t, a_n)$, is taken at some chosen scale, $a_n$, by the convolution:

$$W(t, a_n) = \left( \frac{1}{a_n} \right)^{1/2} \int_{-\infty}^{\infty} g \left( \frac{t'-t}{a_n} \right) f(t') dt'$$  \hspace{1cm} (5)$$

where $f(t')$ is our time series. Here we have chosen the ‘square root of $a_n$’ normalization to force the wavelet to have unit energy at all scales (Torrence and Compo, 1998). This decision was made on the basis of testing a synthetic Brownian walk time series and requiring that the normalization factor give a slope of two in log-log frequency space.

Other wavelet forms are possible, each having particular strengths and drawbacks. The Mexican hat wavelet is relatively wide in the spectral domain, meaning that at a given scale the Mexican hat responds to a range of frequencies in the time series. The Morlet wavelet, a sine wave modulated by a Gaussian, is narrow in frequency space, giving it finer spectral resolution than the Mexican hat wavelet. In general, there is a trade-off between spatial and spectral resolution inherent in the choice of wavelet (Torrence and Compo, 1998). The broad spectral response associated with the Mexican hat wavelet is especially problematic when analyzing time series with dominant periodic components. In these cases one should chose the Morlet wavelet, or simply Fourier spectral analysis.

Because time series that are self-affine exhibit variance over a broad range of frequencies, we wish to scale the wavelet to address this range efficiently. Thus we scale with powers of two according to the formula:

$$a_n = a_0 2^n, n = 0,1,2,3...N.$$  \hspace{1cm} (6)$$
Where $a_0$ is the minimum wavelet scale, chosen to be the data resolution of the time series.

Much debate has centered on determining the effective wavelength represented by the Mexican hat wavelet at different scales. The effective wavelength is the Fourier wavelength corresponding to the scale value $a_n$. Several different techniques for calculating this quantity have been suggested in the literature. *Torrence and Compo* (1998) find the effective wavelength empirically by convoluting cosine waves of specified wavelength to determine the greatest response. *Weissel* (unpublished, 2000) explains another method in which the centroid of the wavelet in Fourier space gives the effective wavelength. Here we use the former technique, arriving at the conversion:

$$\lambda_n \approx \sqrt{2\pi}a_n$$

which is approximately twice that determined by Weissel (unpublished, 2000). This discrepancy will not be resolved here.

Referring to equation (5) above, we assign units of time to the variables $t$, $t'$ and $a_n$. Letting $f(t')$ have units of length, the wavelet transform, $W(t, a_n)$ thus has units of length – time$^{1/2}$. In our analysis we will utilize the variance $V_n$ of the wavelet transform at each scale, which has units of length$^2$- time$^1$. Note that these are the same units as the power spectral density of the data series would have. In fact, $V_n$ is entirely analogous to $S$, the power spectral density. Thus we can rewrite equation (1):

$$V_n \propto \lambda_n^\beta$$

We are using wavelet analysis as an alternative to Fourier power spectral analysis to avoid some of the pitfalls inherent in applying the FFT to a non-stationary series.
Since we are working with discrete time series, we need a discrete version of the wavelet transform for implementation on the computer. Here we consider a discrete time variable \( t_k \) with \( N \) points, adjacent points separated in time by \( a_0 \). Converting the integral in equation (5) to a sum we have

\[
W(t_k, a_n) = \sqrt{\frac{a_0}{2^n}} \sum_{k=0}^{N-1} g \left[ \frac{(t_k - t_{k'})}{a_n} \right] f(t_{k'}). \tag{9}
\]

This is a convolution that can be performed by many standard mathematical packages such as Matlab or IDL\(^1\).

\(^1\) This operation can be performed much more efficiently in Fourier space. For details see Torrence and Compo (1998).