FINITE DIFFERENCE MODELING OF INFRASOUND PROPAGATION TO LOCAL AND REGIONAL DISTANCES

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ABSTRACT

The finite difference (FD) method yields the solution to a discretized version of the full acoustic wave equation for arbitrarily complex media. It is a full spectrum approach and is thus reliable at all angles of propagation, including backscatter. This offers an advantage over other standard propagation methods in wide use, as it allows for accurate computation of acoustic energy levels in the case where significant scattering can occur near the source, such as may happen for an explosion near the surface, or underground. This fits in with nuclear monitoring goals, in that it allows for an improved understanding of the generation and propagation of infrasound energy from arbitrary sources, including underground and near-surface explosions.

Two types of FD methods of solving the acoustic wave equation are presented in this paper. The first is a finite difference frequency domain (FDFD) method, applied in cylindrical coordinates to simulate the effects of a point source in an azimuthally symmetric medium. The second is a finite difference time domain (FDTD) approach including the effects of both gravity and wind, applied in two-dimensional Cartesian coordinates. In this paper equations are developed for the FDTD approach where both wind and gravity effects are considered.

It is shown that the FD approach can be used to solve for sound intensities in arbitrarily complex models that may include high material contrasts and arbitrary topography. In this paper, results of FDTD and FDFD approaches are compared for the case of a shallow underground source, for a boundary with significant topography. The effects of wind and gravity on the solution are examined.

OBJECTIVES

An FDTD method of numerically synthesizing infrasound energy in a realistic environment is sought. The FDTD method yields the solution to a discretized version of the full acoustic wave equation for arbitrarily complex media. It is a full spectrum approach and is thus reliable at all angles of propagation, including backscatter. This offers an advantage over other standard propagation methods in wide use, as it allows for accurate computation in cases where significant scattering can occur near the source, such as may happen for an explosion near the surface or underground. The effects of wind or gravity on infrasound propagation may also be incorporated into the infrasound propagation problem with relative ease using finite difference techniques. This fits in with nuclear monitoring goals, in that successful completion of this project will allow for an improved understanding of the generation and propagation of infrasound energy in arbitrarily complex environments.

In this paper, both an FDFD approach and an FDTD approach are used to solve for the infrasound signals generated by an underground source. A method of incorporating wind and gravity are outlined for the FDFD approach.

RESEARCH ACCOMPLISHED

An FDTD method of numerically acousto-gravity waves in a windy environment has been developed. The equations governing infrasound propagation are derived below for the time domain; their incorporation into an FDTD method are explained in the next subsection. Examples are given in the final sub-section and compared to the frequency domain method.

Low frequency wave equations for a fluid in motion

In the absence of viscosity, the equations governing propagation of sound in the atmosphere are the conservation of momentum,

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla P + \mathbf{F}, \tag{1}$$

conservation of mass

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0, \tag{2}$$

and the equation of state

$$\frac{DP}{Dt} = C^2 \frac{D\rho}{Dt}, \tag{3}$$

(Gill, 1982; Ostashev *et al.*, 2005). These equations relate the velocity $\mathbf{V}$, the pressure P, and the density ρ. The external forces acting upon the medium are denoted by $\mathbf{F}$, and C is related to the adiabatic sound speed. At low frequencies, the gravitational force $\mathbf{F} = -\rho \mathbf{g}$ must be included, where

$$\mathbf{g} = [0, 0, 9.8] \ m/s^2 \tag{4}$$

indicates gravitational acceleration; the negative sign on the force indicates that a downward force acts upon a positive density fluctuation caused by the propagating sound wave.

The convective derivative (also known as the Lagrangian derivative) $\frac{D}{Dt}$ is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla). \tag{5}$$

The derivative on the left represents the change with time in a reference frame moving with the fluid. The first term on the right side of Equation (5) represents the change at a point fixed in space. The second term represents the change as the observer moves with the fluid at the velocity **V**, and is called the advective term. Generally, quantities are expressed in terms of a fixed point in space in order to compare computational results with observations made at stationary sensors.

The propagation of sound waves in the atmosphere introduce fluctuations in the pressure, density, and velocity fields. The standard procedure in solving Equations ~(1)–(3) is to consider a solution of the form

$$P = p_o + p_s; \quad \rho = \rho_o + \rho_s; \quad \mathbf{V} = \mathbf{w} + \mathbf{v}; \quad C^2 = c^2 + (c^2)'; \tag{6}$$

where p_o, ρ_o, and **w** are the ambient solutions in the absence of fluctuations p_s, ρ_s, and **v** due to the propagating sound wave. The adiabatic sound speed is given by *c*, and $(c^2)'$ is the perturbation in the squared sound speed caused by the passage of the sound wave (Ostashev et.al., 2005). In what follows, **w** denotes wind velocity profile, and **v** denotes the acoustic particle velocity associated with the sound wave. Waveforms are derived by computing the pressure perturbations, p_s, as a function of time.

To obtain first order linear equations feasible for implementation in an FDTD method, it is assumed that the ambient field is in equilibrium, and that the sound wave pressure is very small in comparison with the ambient pressure. The former assumption implies that Eqs. (1-3) are satisfied in the absence of a propagating sound wave so that zeroth order terms cancel when Eqs. (6) are substituted into Equations (1)-(3). The latter assumption means that second and higher order terms cancel. Retaining only linear terms, Equation (1) becomes

$$\rho_o\Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}\Big) + \rho_s\Big(\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}\cdot\nabla\mathbf{w}\Big) = -\nabla p_s - \rho_s\mathbf{g}, \tag{7}$$

where the external gravitational force has been included.

Under the assumption that winds are steady, *i.e.*, $\partial\mathbf{w}/\partial t = 0$, and there is no variation along the direction of the wind, *i.e.*, $\mathbf{w}\cdot\nabla\mathbf{w} = 0$, the equation governing acoustic particle velocity reduces to

$$\rho_o\Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}\Big) = -\nabla p_s - \rho_s\mathbf{g}. \tag{8}$$

Combining Eqs. (2) and (3), and inserting Eqs. (6) yields the following equation for the pressure fluctuations due to the propagating sound wave:

$$\frac{\partial p_s}{\partial t} + \mathbf{w}\cdot\nabla p_s + \mathbf{v}\cdot\nabla p_0 = -\rho c^2\nabla\cdot\mathbf{v} - \Big(\rho_o(c')^2 + \rho_s c^2\Big)\nabla\cdot\mathbf{w} \tag{9}$$

The hydrostatic equation $\nabla p_o = -\rho_o\mathbf{g}$ holds under the assumption of steady state and range-invariant winds. Furthermore, Ostashev *et al.*(2005) states that the divergence term $\nabla\cdot\mathbf{w}$ may be ignored to order $(w/c)^2$; under this assumption Eq. (9) becomes

$$\frac{\partial p_s}{\partial t} + \mathbf{w}\cdot\nabla p_s = \rho_o\mathbf{v}\cdot\mathbf{g} - \rho_0 c^2\nabla\cdot\mathbf{v}, \tag{10}$$

where the gravity vector is as defined in Eq. (4).

The equation for fluctuations in density may be derived from Eq. (3), the equation of state, as

$$\frac{\partial \rho_s}{\partial t} + \mathbf{v}\cdot\nabla\rho_o + \mathbf{w}\cdot\nabla\rho_s = \frac{1}{c^2}\Big(\frac{\partial p_s}{\partial t} + \mathbf{v}\cdot\nabla p_o + \mathbf{w}\cdot\nabla p_s\Big), \tag{11}$$

under the assumptions that the ambient pressure and density values do not vary with time, and that $(c')^2\mathbf{w}\cdot\nabla\rho_0 = 0$, which is satisfied by horizontal winds, and densities that vary weakly with range. This equation for density variations is similar in form to Eq. 6.14.3 of Gill (1982), with the exception that horizontal winds **w** are included in Eq. 11. Making use of the hydrostatic equation $\nabla p_o = -\rho g$, the above equation may be re-written as

$$\frac{d\rho_s}{dt} - v_z\rho_o N^2/g = c^{-2}\big(\frac{dp_s}{dt}\big), \tag{12}$$

where $g = 9.8m/s^2$ is the vertical component of gravity, v_z is the vertical particle velocity, d/dt represents the time derivative in a reference frame moving with the wind, and the Brunt-Vaisala frequency N - also called the buoyancy frequency - is defined as

$$N^2 = -g\big(\rho_0^{-1}d\rho_o/dz + g/c^2\big). \tag{13}$$

For a stable medium, N is a positive real number equal to the frequency of a parcel in vertical motion. Equation 13 indicates that, for a reference frame moving with the wind, the Brunt-Vaisala frequency is unaltered. However, measurements are made in a stationary reference frame, indicating that the winds control the measured velocity of the gravity waves. The buoyancy frequency depends on the vertical gradient of the ambient density. From the hydrostatic equation $p_o(z) = -\rho_o g$ and the ideal gas law $\delta p_o/\delta z = \rho_o R\,T$, where R is a gas constant and T is the temperature in degrees Kelvin, the ambient density is derived as $\rho_o(z) = \rho_o(0)\exp(-g\,z/RT)/RT$. The vertical derivative of the density may be derived from this relation.

Equations 8, 10, and 11 form a complete set of first order linear equations for the pressure, acoustic particle velocity and density fluctuations associated with a propagating infrasound wave in a windy medium, valid under the assumptions stated previously. To summarize, these equations are valid under the assumptions that the ambient pressure, density, and wind fields do not vary with time, the winds are horizontal, and vary only with altitude. That is, the wind must be range-invariant.

Finite difference modeling of infrasound in a windy environment

Here, an FDTD method is outlined for infrasound propagation in a windy, stratified medium. That is, it is assumed that the wind blows horizontally and is invariant along range. The densities, and sound velocities vary much more gradually in the horizontal direction than vertically. The method is applied to a 2-D model, that is, the source is assumed to be an infinite line source. The method resembles an FDTD method for sound waves in a windy atmosphere described in detail by Ostashev et.al. (2005), but with the addition of gravity.

For a 2-D model in the x-z plane with horizontal wind speed w_x, Eqs. (8) becomes

$$\frac{\partial v_x}{\partial t} = -w_x \frac{\partial v_x}{\partial x} - v_z \frac{\partial w_x}{\partial z} - \rho_o^{-1}\Big(\frac{\partial p_s}{\partial x}\Big) \qquad (14a)$$

$$\frac{\partial v_z}{\partial t} = -w_x \frac{\partial v_z}{\partial x} - \rho_o^{-1}\Big(\frac{\partial p_s}{\partial z} + \rho_s g\Big). \qquad (14b)$$

Eq. (10) becomes

$$\frac{\partial p_s}{\partial t} = -w_x \frac{\partial p_s}{\partial x} + \rho_o v_z g - \rho_o c^2\Big(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\Big) \qquad (15)$$

and Eq. (11) becomes

$$\frac{\partial \rho_s}{\partial t} = -v_z \frac{\partial \rho_o}{\partial z} - w_x \frac{\partial \rho_s}{\partial x} + \frac{1}{c^2}\Big(\frac{\partial p_s}{\partial t} - v_z \rho_o g + w_x \frac{\partial p_s}{\partial x}\Big), \qquad (16)$$

The static sound speed c and ambient density ρ_o may vary with both altitude and range; the wind speed w_x varies only with altitude. Equations. 14–16 are in a form suitable to computation by FDTD techniques. Note that for $w_x = 0$, that is, zero wind velocity, and g=0, these equations reduce to the usual equations for acoustic propagation in a static medium (e.g., Botteldoren, 1994).

The finite difference (FD) method relies on replacing linear partial differential equations by a set of discrete equivalents. Field solutions are then computed over a discrete set of nodes that comprise the spatial grid. Figure 1 indicates how field variables are defined and how the medium is discretized for the staggered grid method, initially developed by Yee (1966). The model is decomposed into a set of discrete cells of dimension $\Delta x \, x \, \Delta z$. The sound speed c and ambient density ρ_o are uniform over a given cell, but may vary from cell to cell. Pressure and gravity nodes are defined at the center of each cell and the velocity variables are located midway between the pressure nodes. The staggered grid formulation increases the accuracy of the FD solution, since central differences are used to compute the discrete derivatives (Taflove and Hagness, 2000). The locations of the vertical velocity nodes are defined in such a way as to allow a rigid surface (v_z=0) to be defined at the bottom of the model.

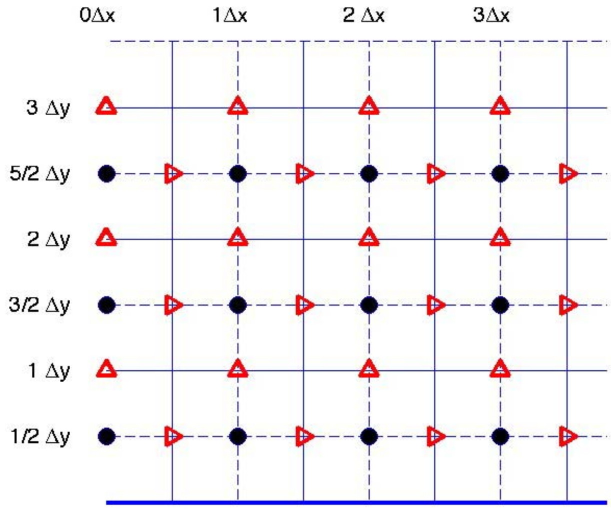

Figure 1. **The finite difference model is decomposed into a set of discrete cells, indicated by the solid lines, each with uniform velocity c and ambient density ρ_o. The acoustic pressure and density perturbations are defined by the nodes at the center of each cell, indicated by the filled circles. The locations of the horizontal velocity v_r (triangles pointing right) and vertical velocity variables v_z (triangles pointing up) are defined on a spatially staggered grid, as shown.**

Typically, in a nonmoving medium, the acoustic velocities and pressures are computed in a leap-frog manner (Yee, 1966), thus the velocities and pressures are computed at alternating time-steps, and the fields from the previous time step are overwritten. Without the wind terms, the governing equations (Equations 14–16) indicate that time derivatives in the acoustic particle velocities depend on the spatial derivatives of the pressure variables, and vice versa. However, with the inclusion of advection terms, first order derivatives in space and time must be computed simultaneously. Various numerical implementations have been suggested for the computation of the FDTD equations for sound propagation in windy environments (Blumrich and Heinmann, 2002; Van Renterghem and Botteldoren, 2003; Ostashev et.al., 2005). Here, the method of Ostashev et.al. (2005) is followed. That is, pressure and velocity fields are over saved over two time steps so that time derivatives may be computed using central differences. Refer to Ostashev et.al., (2005) for further detail.

A method of applying FD methods in the frequency domain, in cylindrical coordinates, for a point source in an azimuthally symmetric model was developed by de Groot-Hedlin (2006).

Comparison of FDFD and FDTD solutions in a model with topography

In this subsection, comparisons are made between FDFD and FDTD results for a shallow underground source in a medium with topography. The source is embedded within a medium with a sound speed 2 km/s and a density of 2000 k/m3. The sound speed within the air is shown in Figure 2.

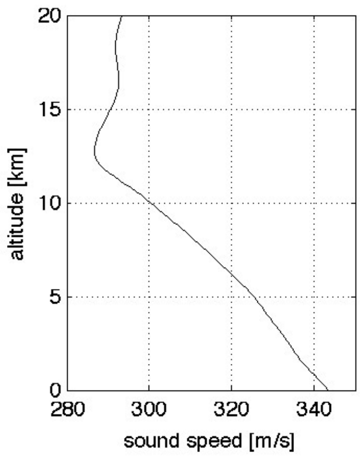

Figure 2. Sound velocity profile within the air. This sound velocity profile is taken from a G2S model for the Mt. St. Helens region for March 9, 2005. The density decreases with altitude by slightly over an order of magnitude over this altitude scale.

The sound velocity profile shows a decrease in sound speed with altitude for the first 10 km; this corresponds to a typical sound velocity profile in which the temperature decreases with altitude. The effect is to cause some deflection of sound upwards, away from the surface.

The finite difference time domain method is applied in Cartesian coordinates, thus the model is invariant along a direction perpendicular to the x-z plane. The source is thus a line source.

This model features a broad, symmetric peak, with a highest altitude of 2 km. The source was located directly beneath the peak at an altitude of 1.2 km, i.e., a distance of 0.9 km below the ground surface. The center frequency of the source was 0.5 Hz. “Receivers” were located at intervals of 5 km from each side of the peak at distances from 5 km to 30 km from the center of the model.

Several “snapshots” of the acoustic pressure are shown in Figure 3. As indicated, the pressure propagates quickly through the ground then couples to the air. A later arrival corresponds to acoustic energy that couples to the air near the source and propagates outward from there. Thus there should be two main arrivals at each receiver, a first one corresponding to acoustic energy propagating through the earth, coupling to the air near the receiver, and a second corresponding to coupling to the air near the source. The traces corresponding to this model are shown in Figure 4. For this model, ray-based propagation methods which rely on a high frequency approximation indicate that acoustic energy is refracted upward, away from the ground so that the more distant receivers lie in a shadow zone. These results thus indicate the extent to which synthesis of the whole waveform is required.

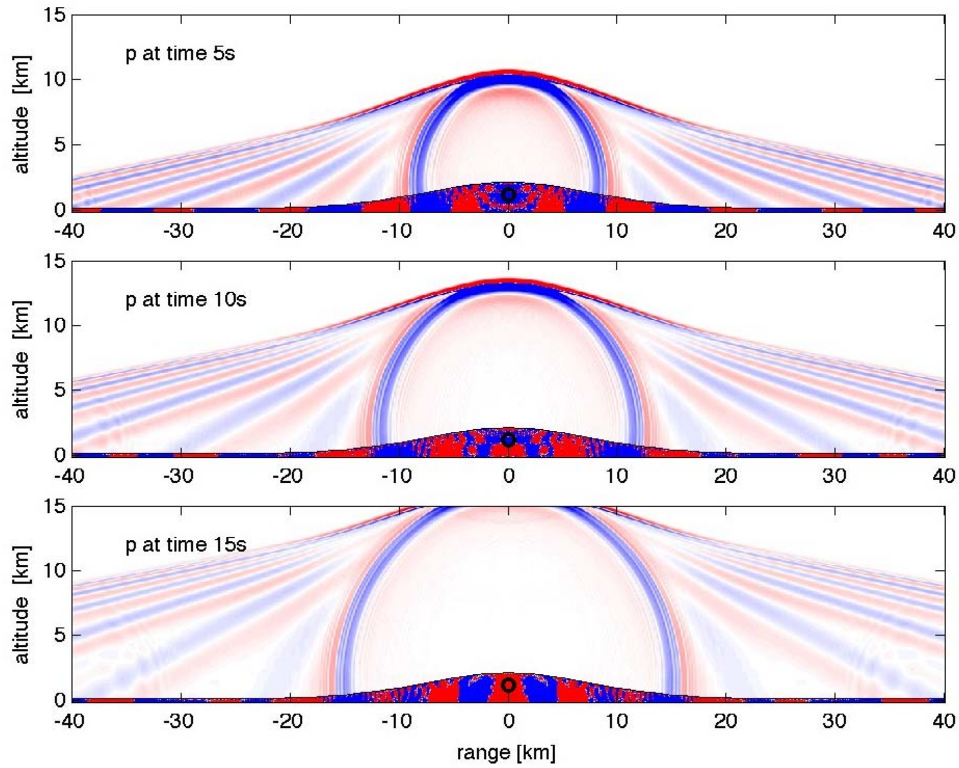

Figure 3. "Snapshots" of the acoustic pressure emanating from a 0.5 Hz source at 0.9km below the central peak at the time points 5 s, 10 s, and 15 s. An identical color scale is shown for each plot.

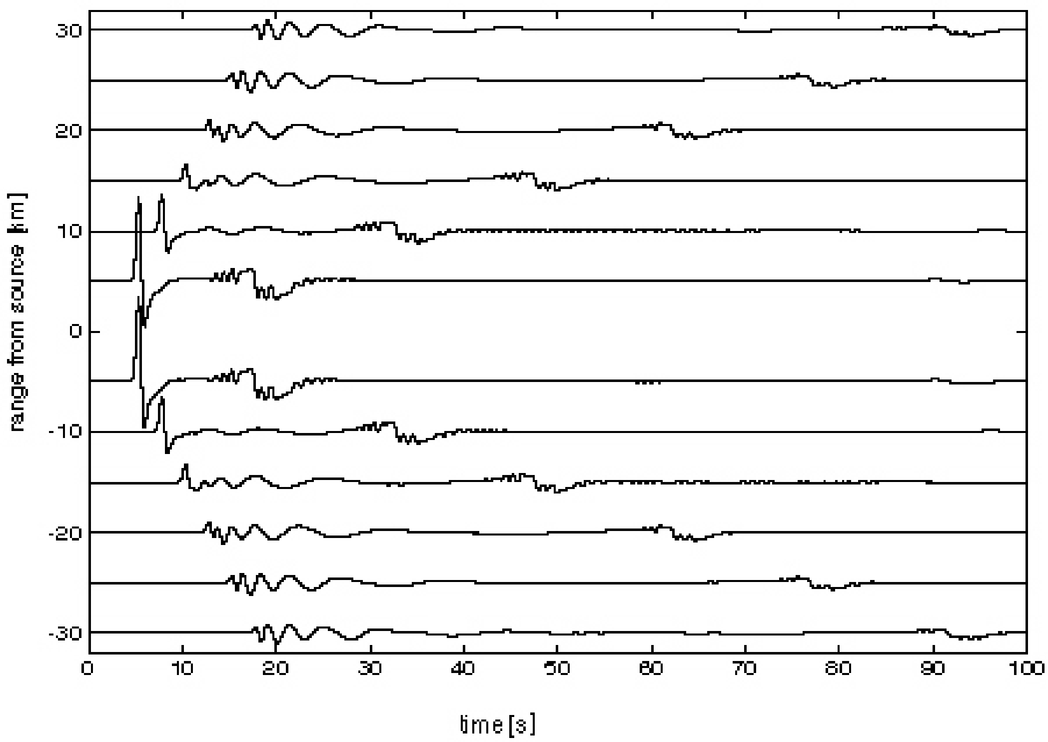

Figure 4. Waveforms at distances of 5 km to 30 km from the peak.

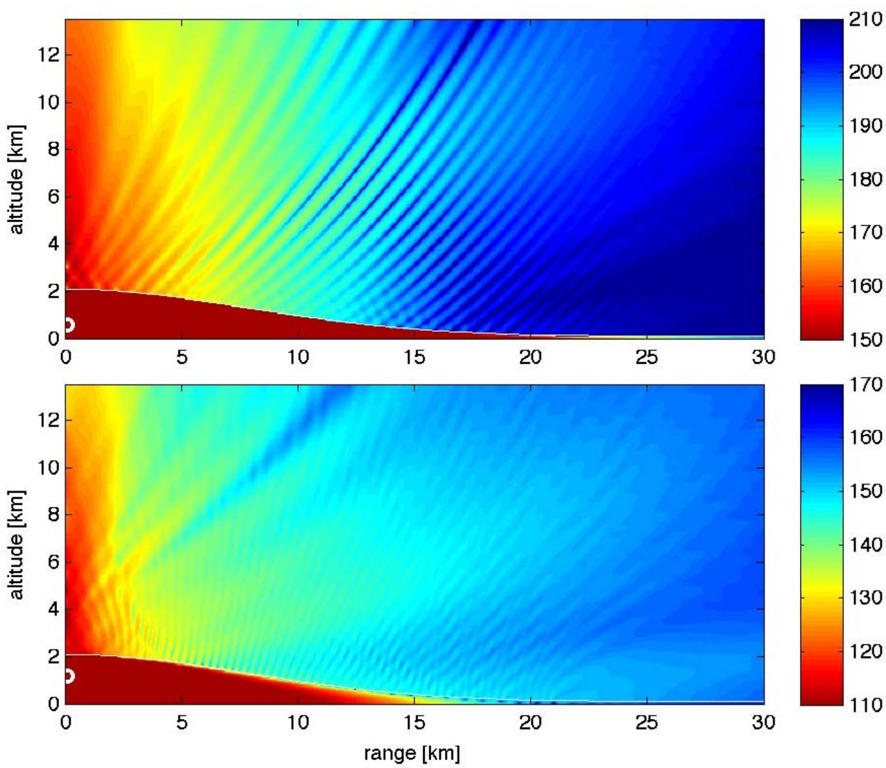

Figure 5. Transmission loss results derived from finite difference frequency domain modeling of (top) a 0.5Hz source at 0.6km, i.e., 1.5km below the surface and (bottom) for the same source at 1.2km. The second source is equivalent to the one shown for the FDTD simulation. Note the difference in colorscales for the two source depths.

For the finite difference time domain results, there was no discernable difference in the results with or without gravity, as gravitational results become significant at much lower frequencies. Incorporation of a realistic wind profile showed only minor asymmetry in the waveform results on either side of the peak. Wind and gravity have not yet been incorporated into finite difference frequency domain modeling.

CONCLUSIONS AND RECOMMENDATIONS

Equations have been developed to incorporate the effects of both wind and gravity into an FDTD modeling method. FDFD methods of incorporating these effects are less robust as they require the inversion of large matrices.

It has been demonstrated that a full waveform modeling technique is required to model infrasound generated by an underground source. Further verification of the FD method is needed for sources within a fraction of a wavelength from a boundary with a strong impedance contrast.

REFERENCES

R. Blumrich, and D. Heinmann (2002). A linearized Eulerian sound propagation model for studies of complex meteorological effects, *J. Acoust. Soc. Am.* 112: 446–455.

Botteldoren, D., 1994, Acoustical finite-difference time-domain simulation in a quasi-Cartesian grid, *J. Acoust. Soc. Am.* 95: 2204–2212.

de Groot-Hedlin, C., (2006). Finite difference methods for acoustic and acousto-gravity wavefields: application to low frequency infrasound propagation, in *Proceeding of the 28th Seismic Research Review: Ground-Based Nuclear Explosion Monitoring Technologies*, LA-UR-06-5471, Vol. 2, pp. 893–899.

Gill, A. E. (1982). *Atmosphere-Ocean Dynamics*, Academic Press, San Diego, CA.

Ostashev, V.E., D.K. Wilson, L. Liu, D.F. Aldridge, N.P. Symons, D. Marlin (2005). Equations for finite-difference, time-domain simulation of sound propagation in moving inhomogeneous media and numerical implementation, *J. Acoust. Soc. Am.*, 117, 503–517.

Van Renterghem, T., and D. Botteldoren, (2003). Numerical simulation of the effect of trees on downwind noise barrier performance, *Acust. Acta. Acust.* 89: 764–778.

Yee, K.S, 1966, Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media, *IEEE Trans. Antennas and Propagation* 14: 302–307.