Lecture 11

Vector Spaces
and
Singular Value Decomposition

Syllabus

Lecture 01	Describing Inverse Problems
Lecture 02	Probability and Measurement Error, Part 1
Lecture 03	Probability and Measurement Error, Part 2
Lecture 04	The L_2 Norm and Simple Least Squares
Lecture 05	A Priori Information and Weighted Least Squared
Lecture 06	Resolution and Generalized Inverses
Lecture 07	Backus-Gilbert Inverse and the Trade Off of Resolution and Variance
Lecture 08	The Principle of Maximum Likelihood
Lecture 09	Inexact Theories
Lecture 10	Nonuniqueness and Localized Averages
Lecture 11	**Vector Spaces and Singular Value Decomposition**
Lecture 12	Equality and Inequality Constraints
Lecture 13	L_1 , L_∞ Norm Problems and Linear Programming
Lecture 14	Nonlinear Problems: Grid and Monte Carlo Searches
Lecture 15	Nonlinear Problems: Newton's Method
Lecture 16	Nonlinear Problems: Simulated Annealing and Bootstrap Confidence Intervals
Lecture 17	Factor Analysis
Lecture 18	Varimax Factors, Empirical Orthogonal Functions
Lecture 19	Backus-Gilbert Theory for Continuous Problems; Radon's Problem
Lecture 20	Linear Operators and Their Adjoints
Lecture 21	Fréchet Derivatives
Lecture 22	Exemplary Inverse Problems, incl. Filter Design
Lecture 23	Exemplary Inverse Problems, incl. Earthquake Location
Lecture 24	Exemplary Inverse Problems, incl. Vibrational Problems

Purpose of the Lecture

View **m** and **d** as points in the space of model parameters and data

Develop the idea of transformations of coordinate axes

Show how transformations can be used to convert a weighted problem into an unweighted one

Introduce the Natural Solution and the Singular Value Decomposition

Part 1

the spaces of model parameters and data

what is a vector?

algebraic viewpoint

a vector is a quantity that is manipulated

(especially, multiplied)

via a specific set of rules

geometric viewpoint

a vector is a direction and length

in space

what is a vector?

algebraic viewpoint

column-

a vector is a quantity that is manipulated

(especially, multiplied)

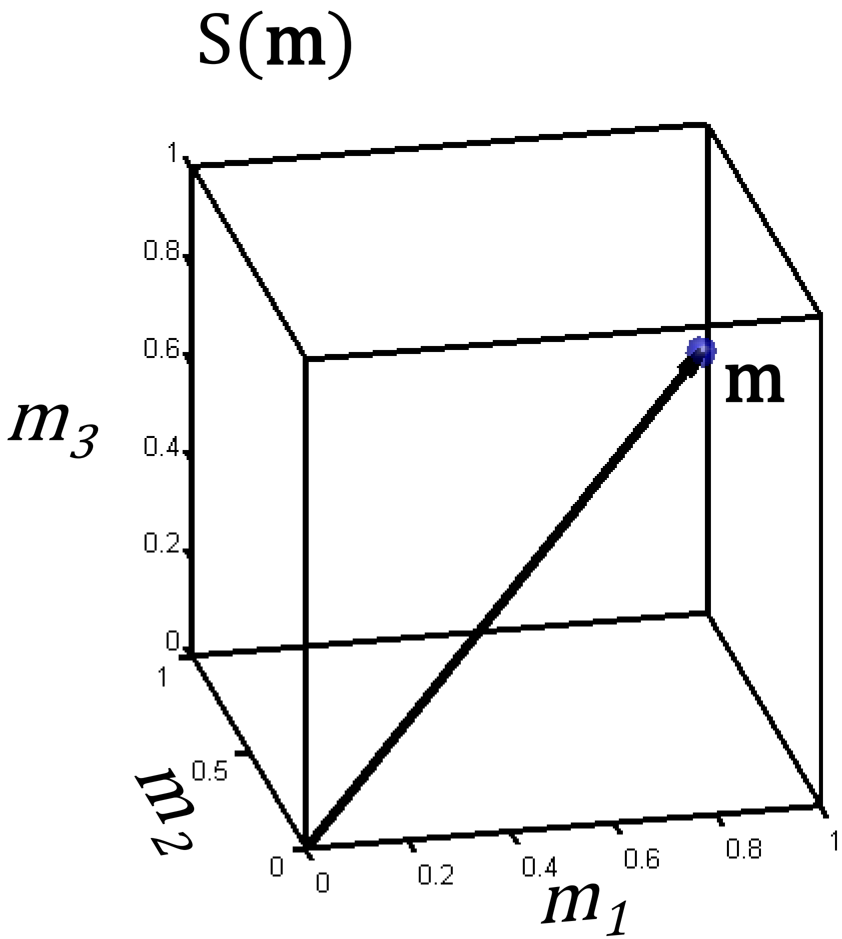
via a specific set of rules

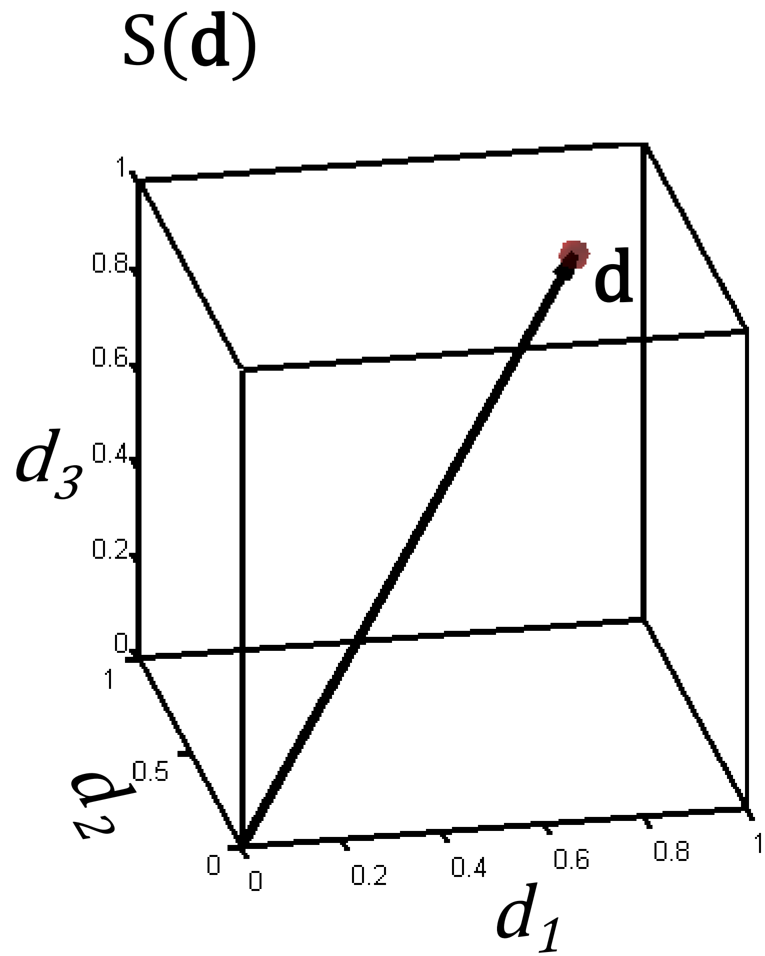
geometric viewpoint

a vector is a direction and length

in space

in our case, a space of very high dimension


S(**m**)
m
m_3
m_2
m_1
1
0.8
0.6
0.4
0.2
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1


S(**d**)
d
d_3
d_2
d_1
1
0.8
0.6
0.4
0.2
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1

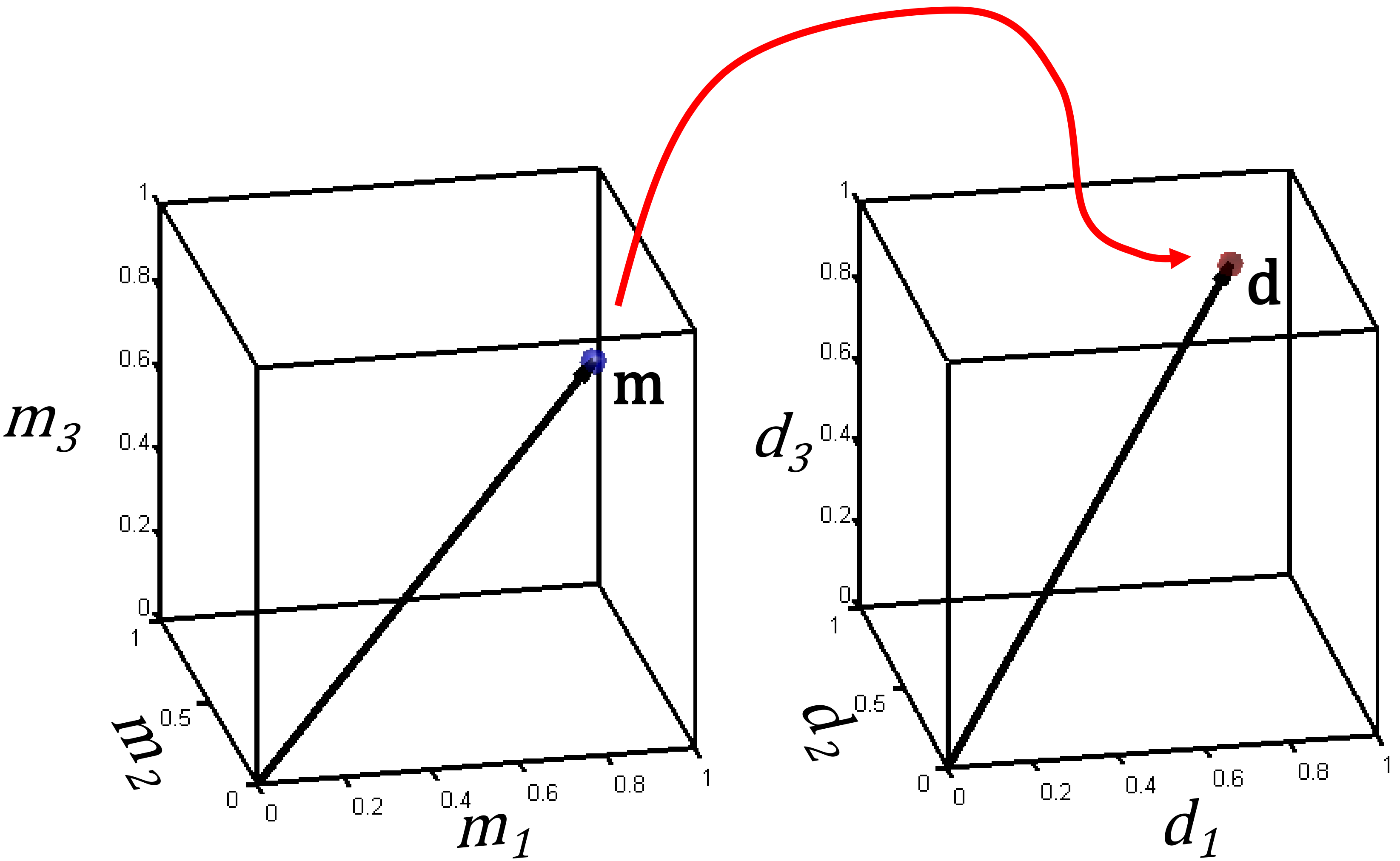
forward problem

$$\mathbf{d} = \mathbf{G}\mathbf{m}$$

maps an **m** onto a **d**

maps a point in S(**m**) to a point in S(**d**)

Forward Problem: Maps S(**m**) onto S(**d**)



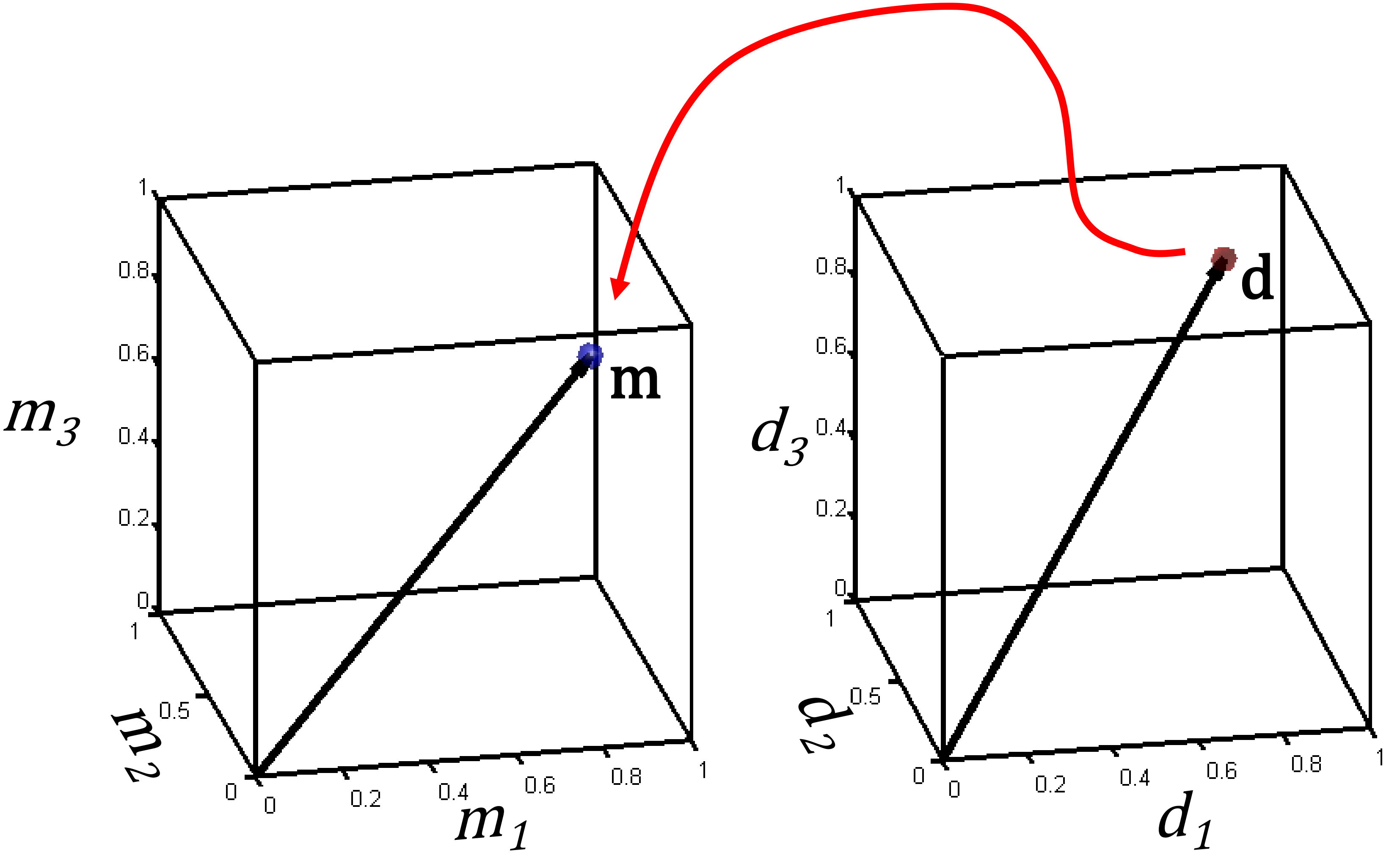
inverse problem

$$\mathbf{m} = \mathbf{G}^{-g}\mathbf{d}$$

maps a **d** onto an **m**

maps a point in S(**m**) to a point in S(**d**)

Inverse Problem: Maps S(**d**) onto S(**m**)



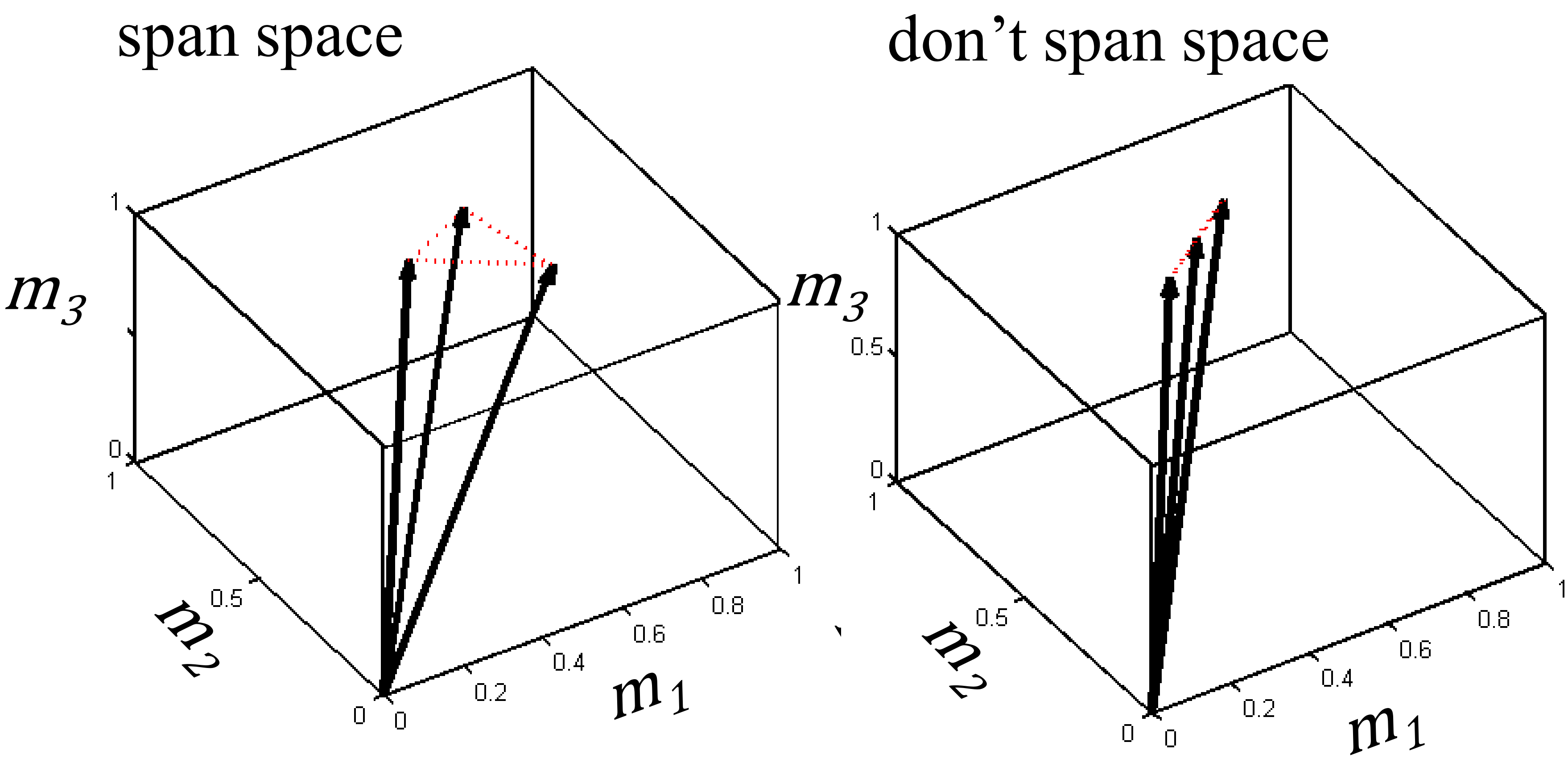
Part 2

Transformations of coordinate axes

coordinate axes are arbitrary

given M linearly-independent *basis* vectors $\mathbf{m}^{(i)}$

we can write any vector $\mathbf{m}^*$ as ...


span space
don’t span space
m3
m2
m1
m3
m2
m1
1
0
0.5
0.2
0.4
0.6
0.8

... as a linear combination of these *basis vectors*

$$\mathbf{m}^* = \sum_{i=1}^{M} \alpha_i \mathbf{m}^{(i)}$$

... as a linear combination of these *basis vectors*

$$\mathbf{m}^* = \sum_{i=1}^{M} \alpha_i \mathbf{m}^{(i)}$$

components of **m*** in new coordinate system

$$\mathbf{m}_i^{*\prime} = \alpha_i$$

might it be fair to say
that the components of a vector
are a column-vector
?

$$\mathbf{m}^* = \sum_{i=1}^{M} \alpha_i \mathbf{m}^{(i)}$$

$$m_i^* = \sum_{i=1}^{M} v_j^{(i)} m_j^{*\prime} = \sum_{i=1}^{M} M_{ij} m_j^{*\prime}$$

matrix formed from
basis vectors
$M_{ij} = v_j^{(i)}$

transformation matrix $\mathbf{T}$

$$\mathbf{m}' = \mathbf{T}\mathbf{m} \quad \text{and} \quad \mathbf{m} = \mathbf{T}^{-1}\mathbf{m}'$$

transformation matrix $\mathbf{T}$

$$\mathbf{m}' = \mathbf{T}\mathbf{m} \quad \text{and} \quad \mathbf{m} = \mathbf{T}^{-1}\mathbf{m}'$$

same vector
different components

Q: does **T** preserve "length" ?
(in the sense that $\mathbf{m}^{\mathrm{T}}\mathbf{m} = \mathbf{m}'^{\mathrm{T}}\mathbf{m}'$)

$$L = \mathbf{m}^{\mathrm{T}}\mathbf{m} = \{\mathbf{T}^{-1}\mathbf{m}'\}^{\mathrm{T}}\{\mathbf{T}^{-1}\mathbf{m}'\} = \mathbf{m}'^{\mathrm{T}}\{\mathbf{T}^{-1\mathrm{T}}\mathbf{T}^{-1}\}\mathbf{m}'$$

$$\mathbf{T} \text{ so that } \{\mathbf{T}^{-1\mathrm{T}}\mathbf{T}^{-1}\} = \mathbf{I}$$

A: only when $\mathbf{T}^{\mathrm{T}} = \mathbf{T}^{-1}$

transformation of the model space axes

$$\mathbf{d} = \mathbf{Gm} = \mathbf{GIm} = [\mathbf{GT}_{\mathrm{m}}^{-1}]\ [\mathbf{T}_{\mathrm{m}}\mathbf{m}] = \mathbf{G'm'}$$

$$\left.\begin{aligned} \mathbf{d} &= \mathbf{Gm} \\ \mathbf{d} &= \mathbf{G'm'} \end{aligned}\right\}$$

same equation
different coordinate
system for **m**

transformation of the data space axes

$$\mathbf{d}' = \mathbf{T}_\mathrm{d}\mathbf{d} = [\mathbf{T}_\mathrm{d}\mathbf{G}]\,\mathbf{m} = \mathbf{G}''\mathbf{m}$$

$$\left.\begin{array}{l} \mathbf{d} = \mathbf{G}\mathbf{m} \\ \mathbf{d}' = \mathbf{G}''\mathbf{m} \end{array}\right\}$$

same equation
different coordinate
system for **d**

transformation of both data space and model space axes

$$\mathbf{d}' = \mathbf{T}_{\mathrm{d}}\mathbf{d} = [\mathbf{T}_{\mathrm{d}}\mathbf{G}\mathbf{T}_{\mathrm{m}}^{-1}]\,[\mathbf{T}_{\mathrm{m}}\mathbf{m}] = \mathbf{G}'''\mathbf{m}'$$

$$\left.\begin{aligned} \mathbf{d} &= \mathbf{G}\mathbf{m} \\ \mathbf{d}' &= \mathbf{G}'''\mathbf{m}' \end{aligned}\right\}$$

same equation
different coordinate
systems for **d** and **m**

Part 3

how transformations can be used to convert a weighted problem into an unweighted one

when are transformations useful ?

remember this?

minimize: $E + L = \mathbf{e}^{\mathrm{T}}\mathbf{W}_e\,\mathbf{e} + \mathbf{m}^{\mathrm{T}}\mathbf{W}_m\,\mathbf{m}$

when are transformations useful ?

remember this?

minimize: $E + L = \mathbf{e}^{\mathrm{T}}\mathbf{W}_e\,\mathbf{e} + \mathbf{m}^{\mathrm{T}}\mathbf{W}_m\,\mathbf{m}$

massage this into a pair
of transformations

$$\mathbf{m}^{T}\mathbf{W}_{m}\mathbf{m}$$

$$\mathbf{W}_{m}=\mathbf{D}^{T}\mathbf{D} \quad \text{or} \quad \mathbf{W}_{m}=\mathbf{W}_{m}^{\frac{1}{2}}\mathbf{W}_{m}^{\frac{1}{2}}=\mathbf{W}_{m}^{\frac{1}{2}T}\mathbf{W}_{m}^{\frac{1}{2}}$$

OK since $\mathbf{W}_{m}$ symmetric

$$\mathbf{m}^{T}\mathbf{W}_{m}\mathbf{m}=\mathbf{m}^{T}\mathbf{D}^{T}\mathbf{D}\mathbf{m}=[\mathbf{D}\mathbf{m}]^{T}[\mathbf{D}\mathbf{m}]$$

$\mathbf{T}_{\mathbf{m}}$

when are transformations useful ?

remember this?

minimize: $E + L = \mathbf{e}^{\mathrm{T}}\mathbf{W}_e\,\mathbf{e} + \mathbf{m}^{\mathrm{T}}\mathbf{W}_m\,\mathbf{m}$

massage this into a pair
of transformations

$$\mathbf{e}^{\mathrm{T}}\mathbf{W}_{\mathrm{e}}\mathbf{e}$$

$$\mathbf{W}_{\mathrm{e}} = \mathbf{W}_{\mathrm{e}}^{1/2}\mathbf{W}_{\mathrm{e}}^{1/2} = \mathbf{W}_{\mathrm{e}}^{1/2\mathrm{T}}\mathbf{W}_{\mathrm{e}}^{1/2}$$

OK since $\mathbf{W}_{\mathrm{e}}$ symmetric

$$\mathbf{e}^{\mathrm{T}}\mathbf{W}_{\mathrm{e}}\mathbf{e} = \mathbf{e}^{\mathrm{T}}\mathbf{W}_{\mathrm{e}}^{1/2\mathrm{T}}\mathbf{W}_{\mathrm{e}}^{1/2}\mathbf{e} = [\mathbf{W}_{\mathrm{e}}^{1/2}\mathbf{m}]^{\mathrm{T}}[\mathbf{W}_{\mathrm{e}}^{1/2}\mathbf{m}]$$

$\mathbf{T}_{\mathbf{d}}$

we have converted weighted least-squares

minimize: $E + L = \mathbf{e}^{\mathrm{T}}\mathbf{W}_e\mathbf{e} + \mathbf{m}^{\mathrm{T}}\mathbf{W}_m\mathbf{m}$

into unweighted least-squares

minimize: $E' + L' = \mathbf{e}'^{\mathrm{T}}\mathbf{e}' + \mathbf{m}'^{\mathrm{T}}\mathbf{m}'$

steps

1: Compute Transformations

$$\mathbf{T}_m = \mathbf{D} = \mathbf{W}_m^{½} \text{ and } \mathbf{T}_e = \mathbf{W}_e^{½}$$

2: Transform data kernel and data to new coordinate system

$$\mathbf{G}''' = [\mathbf{T}_e \mathbf{G} \mathbf{T}_m^{-1}] \quad \text{and } \mathbf{d}' = \mathbf{T}_e \mathbf{d}$$

3: solve $\mathbf{G}'''\ \mathbf{m}' = \mathbf{d}'$ for $\mathbf{m}'$ using unweighted method

4: Transform $\mathbf{m}'$ back to original coordinate system

$$\mathbf{m} = \mathbf{T}_m^{-1} \mathbf{m}'$$

steps

extra work

1: Compute Transformations

$$\mathbf{T}_m = \mathbf{D} = \mathbf{W}_m^{½} \text{ and } \mathbf{T}_e = \mathbf{W}_e^{½}$$

2: Transform data kernel and data to new coordinate system

$$\mathbf{G}''' = [\mathbf{T}_e \mathbf{G} \mathbf{T}_m^{-1}] \quad \text{and } \mathbf{d}' = \mathbf{T}_e \mathbf{d}$$

3: solve $\mathbf{G}''' \, \mathbf{m}' = \mathbf{d}'$ for $\mathbf{m}'$ using unweighted method

4: Transform $\mathbf{m}'$ back to original coordinate system

$$\mathbf{m} = \mathbf{T}_m^{-1} \mathbf{m}'$$

steps

to allow simpler solution method

1: Compute Transformations

$$\mathbf{T}_m = \mathbf{D} = \mathbf{W}_m^{½} \text{ and } \mathbf{T}_e = \mathbf{W}_e^{½}$$

2: Transform data kernel and data to new coordinate system

$$\mathbf{G}''' = [\mathbf{T}_e \mathbf{G} \mathbf{T}_m^{-1}] \text{ and } \mathbf{d}' = \mathbf{T}_e \mathbf{d}$$

3: solve $\mathbf{G}''' \, \mathbf{m}' = \mathbf{d}'$ for $\mathbf{m}'$ using unweighted method

4: Transform $\mathbf{m}'$ back to original coordinate system

$$\mathbf{m} = \mathbf{T}_m^{-1} \mathbf{m}'$$

Part 4

The Natural Solution and the Singular Value Decomposition (SVD)

$$\mathbf{G}\mathbf{m} = \mathbf{d}$$

$$\Downarrow$$

$$\mathbf{G}\left[\mathbf{m}_{\mathrm{p}} + \mathbf{m}_{0}\right] = \left[\mathbf{d}_{\mathrm{p}} + \mathbf{d}_{0}\right]$$

suppose that we could divide up the problem like this ...

$$\mathbf{Gm} = \mathbf{d}$$

$$\mathbf{G}[\mathbf{m}_\mathrm{p} + \mathbf{m}_0] = [\mathbf{d}_\mathrm{p} + \mathbf{d}_0]$$

only $\mathbf{m}_\mathrm{p}$ can affect $\mathbf{d}$

since $\mathbf{Gm}_0 = 0$

$$\mathbf{Gm} = \mathbf{d}$$

$$\mathbf{G}[\mathbf{m}_\mathrm{p} + \mathbf{m}_0] = [\mathbf{d}_\mathrm{p} + \mathbf{d}_0]$$

$\mathbf{Gm}_\mathrm{p}$ can only affect $\mathbf{d}_\mathrm{p}$
since no $\mathbf{m}$ can
lead to a $\mathbf{d}_0$

$$L = \mathbf{m}^{\mathrm{T}}\mathbf{m} = [\mathbf{m}_{\mathrm{p}} + \mathbf{m}_0]^{\mathrm{T}}[\mathbf{m}_{\mathrm{p}} + \mathbf{m}_0] = \mathbf{m}_{\mathrm{p}}^{\mathrm{T}}\mathbf{m}_{\mathrm{p}} + \mathbf{m}_0^{\mathrm{T}}\mathbf{m}_0$$

$$E = [\mathbf{d}_{\mathrm{p}} + \mathbf{d}_0 - \mathbf{G}\mathbf{m}_{\mathrm{p}}]^{\mathrm{T}}[\mathbf{d}_{\mathrm{p}} + \mathbf{d}_0 - \mathbf{G}\mathbf{m}_{\mathrm{p}}] =$$

$$[\mathbf{d}_{\mathrm{p}} - \mathbf{G}\mathbf{m}_{\mathrm{p}}]^{\mathrm{T}}[\mathbf{d}_{\mathrm{p}} - \mathbf{G}\mathbf{m}_{\mathrm{p}}] + \mathbf{d}_0^{\mathrm{T}}\mathbf{d}_0$$

$$L = \mathbf{m}^{\mathrm{T}}\mathbf{m} = [\mathbf{m}_{\mathrm{p}} + \mathbf{m}_0]^{\mathrm{T}}[\mathbf{m}_{\mathrm{p}} + \mathbf{m}_0] = \overbrace{\mathbf{m}_{\mathrm{p}}^{\mathrm{T}}\mathbf{m}_{\mathrm{p}}}^{\text{determined by data}} + \overbrace{\mathbf{m}_0^{\mathrm{T}}\mathbf{m}_0}^{\text{determined by a priori information}}$$

$$E = [\mathbf{d}_{\mathrm{p}} + \mathbf{d}_0 - \mathbf{G}\mathbf{m}_{\mathrm{p}}]^{\mathrm{T}}[\mathbf{d}_{\mathrm{p}} + \mathbf{d}_0 - \mathbf{G}\mathbf{m}_{\mathrm{p}}] =$$

$$\underbrace{[\mathbf{d}_{\mathrm{p}} - \mathbf{G}\mathbf{m}_{\mathrm{p}}]^{\mathrm{T}}}_{\text{determined by } \mathbf{m}_{\mathrm{p}}}[\mathbf{d}_{\mathrm{p}} - \mathbf{G}\mathbf{m}_{\mathrm{p}}]^{\mathrm{T}} + \underbrace{\mathbf{d}_0^{\mathrm{T}}\mathbf{d}_0}_{\text{not possible to reduce}}$$

natural solution

determine $\mathbf{m}_p$ by solving $\mathbf{d}_p\text{-}\mathbf{G}\mathbf{m}_p=0$

set $\mathbf{m}_0=0$

what we need is a way to do

$$\mathbf{Gm} = \mathbf{d}$$

$$\Downarrow$$

$$\mathbf{G}\left[\mathbf{m}_{\mathrm{p}} + \mathbf{m}_0\right] = \left[\mathbf{d}_{\mathrm{p}} + \mathbf{d}_0\right]$$

Singular Value Decomposition (SVD)

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{T}}$$

$N \times N$ matrix of eigenvectors

$$\mathbf{U} = [\mathbf{u}^{(1)} \quad \mathbf{u}^{(2)} \quad \mathbf{u}^{(3)} \quad \dots \quad \mathbf{u}^{\mathrm{N}}]$$

$M \times M$ matrix of eigenvectors

$$\mathbf{V} = [\mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \mathbf{v}^{(3)} \quad \dots \quad \mathbf{v}^{\mathrm{M}}]$$

$\boldsymbol{\Lambda}$ is an $N \times M$ diagonal matrix

singular value decomposition

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{T}}$$

$\mathbf{U}^{\mathrm{T}}\mathbf{U}=\mathbf{I}$ and $\mathbf{V}^{\mathrm{T}}\mathbf{V}=\mathbf{I}$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \end{bmatrix}$$

suppose only p λ's are non-zero

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_\mathrm{p} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

suppose only *p* λ's are non-zero

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{T} \Rightarrow \mathbf{U}_{p}\boldsymbol{\Lambda}_{p}\mathbf{V}_{p}^{T}$$

only first *p* columns of **U**

only first *p* columns of **V**

$$\mathbf{U}_p^T\mathbf{U}_p = \mathbf{I} \quad \text{and} \quad \mathbf{V}_p^T\mathbf{V}_p = \mathbf{I}$$

since vectors mutually pependicular and of unit length

$$\mathbf{U}_p\mathbf{U}_p^T \neq \mathbf{I} \quad \text{and} \quad \mathbf{V}_p\mathbf{V}_p^T \neq \mathbf{I}$$

since vectors do not span entire space

The part of $\mathbf{m}$ that lies in $\mathbf{V}_0$ cannot effect $\mathbf{d}$

$$\mathbf{d} = \mathbf{G}\mathbf{m} = \mathbf{U}_p \mathbf{\Lambda}_p \mathbf{V}_p^T \mathbf{m}$$

since $\mathbf{V}_p{}^T\mathbf{V}_0 = 0$

so $\mathbf{V}_0$ is the model null space

The part of $\mathbf{d}$ that lies in $\mathbf{U}_0$ cannot be affected by $\mathbf{m}$

$$\mathbf{d} = \mathbf{G}\mathbf{m} = \mathbf{U}_p \boldsymbol{\Lambda}_p \mathbf{V}_p^T \mathbf{m}$$

since $\boldsymbol{\Lambda}_p \mathbf{V}_p^T \mathbf{m}$ is multiplied by $\mathbf{U}_p$ and $\mathbf{U}_0 \, \mathbf{U}_p^T = 0$

so $\mathbf{U}_0$ is the data null space

The Natural Solution

$$\mathbf{m}^{\mathrm{est}} = \mathbf{V}_{\mathrm{p}} \mathbf{\Lambda}_{\mathrm{p}}^{-1} \mathbf{U}_{\mathrm{p}}^{\mathrm{T}} \mathbf{d}$$

The part of $\mathbf{m}^{\text{est}}$ in $\mathbf{V}_0$ has zero length

$$\mathbf{V}_0^{\mathrm{T}}\mathbf{m}^{\text{est}} = \mathbf{V}_0^{\mathrm{T}}\mathbf{V}_{\mathrm{p}}\boldsymbol{\Lambda}_{\mathrm{p}}^{-1}\mathbf{U}_{\mathrm{p}}^{\mathrm{T}}\mathbf{d} = \mathbf{0}$$

The error has no component in $\mathbf{U}_\mathrm{p}$

$$\mathbf{U}_\mathrm{p}^\mathrm{T}\mathbf{e} = \mathbf{U}_\mathrm{p}^\mathrm{T}[\mathbf{d} - \mathbf{G}\mathbf{m}^{\mathrm{est}}] = \mathbf{U}_\mathrm{p}^\mathrm{T}\left[\mathbf{d} - \mathbf{U}_\mathrm{p}\boldsymbol{\Lambda}_\mathrm{p}\mathbf{V}_\mathrm{p}^\mathrm{T}\mathbf{V}_\mathrm{p}\boldsymbol{\Lambda}_\mathrm{p}^{-1}\mathbf{U}_\mathrm{p}^\mathrm{T}\mathbf{d}\right]$$

$$= \mathbf{U}_\mathrm{p}^\mathrm{T}\left[\mathbf{d} - \mathbf{U}_\mathrm{p}\mathbf{U}_\mathrm{p}^\mathrm{T}\mathbf{d}\right] = \mathbf{U}_\mathrm{p}^\mathrm{T}\mathbf{d} - \mathbf{U}_\mathrm{p}^\mathrm{T}\mathbf{d} = \mathbf{0}$$

computing the SVD

```
[U, L, V] = svd(G);
lambda = diag(L);
```

determining p

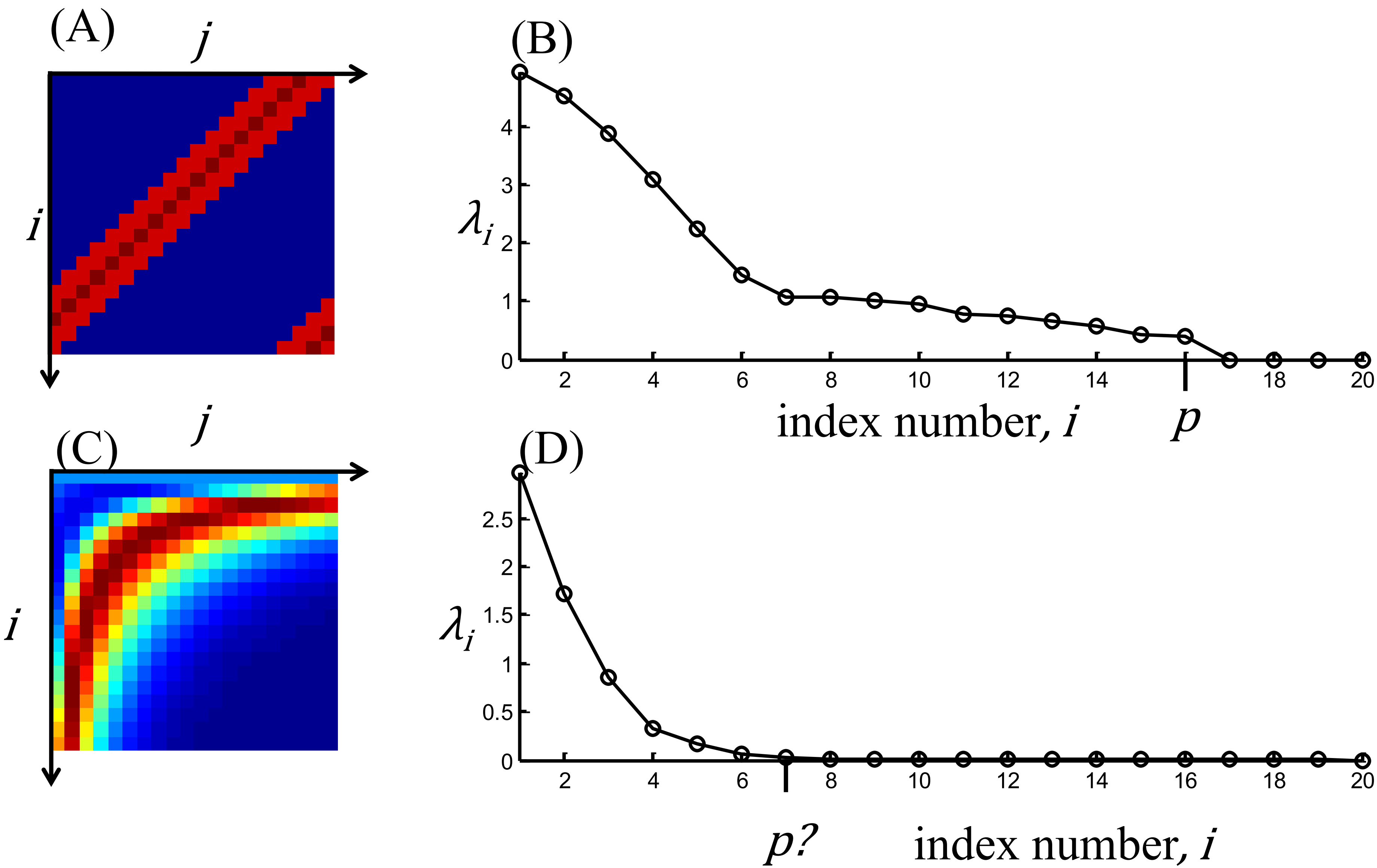
use plot of λ_i vs. i

however

case of a clear division between

$\lambda_i > 0$ and $\lambda_i = 0$

rare

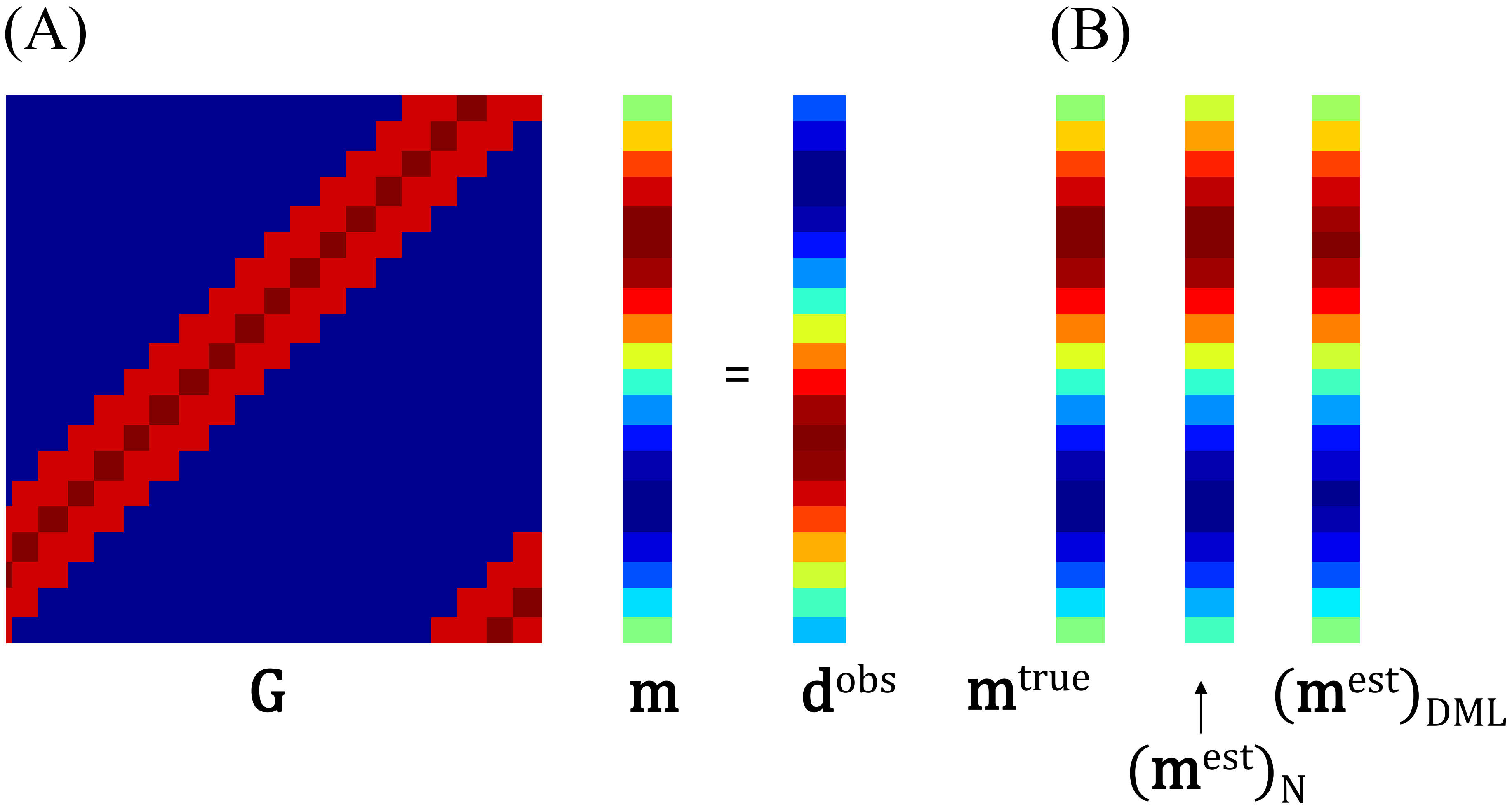

(A)
j
i
(B)
λ_i
0
1
2
3
4
2
4
6
8
10
12
14
18
20
index number, i
p
(C)
j
i
(D)
λ_i
0
0.5
1
1.5
2
2.5
2
4
6
8
10
12
14
16
18
20
p?
index number, i

Natural Solution

```
Up=U(:,1:p);
Vp=V(:,1:p);
lambdap=lambda(1:p);
```

and the model parameters are estimated as

```
mest = Vp*((Up'*dobs)./lambdap);
```


(A)
(B)
G
m
=
$\mathbf{d}^{obs}$
$\mathbf{m}^{true}$
$(\mathbf{m}^{est})_{N}$
$(\mathbf{m}^{est})_{DML}$

resolution and covariance

$$\mathbf{R} = \mathbf{G}^{-g}\mathbf{G} = \{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}\{\mathbf{U}_p\mathbf{\Lambda}_p\mathbf{V}_p^T\} = \mathbf{V}_p\mathbf{V}_p^T$$

$$\mathbf{N} = \mathbf{G}\mathbf{G}^{-g} = \{\mathbf{U}_p\mathbf{\Lambda}_p\mathbf{V}_p^T\}\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\} = \mathbf{U}_p\mathbf{U}_p^T$$

$$[\mathrm{cov}\,\mathbf{m}^{\mathrm{est}}] = \mathbf{G}^{-g}[\mathrm{cov}\,\mathbf{d}]\mathbf{G}^{-gT} =$$

$$= \sigma_d^2\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}^T = \sigma_d^2\mathbf{V}_p\mathbf{\Lambda}_p^{-2}\mathbf{V}_p^T$$

resolution and covariance

$$\mathbf{R} = \mathbf{G}^{-g}\mathbf{G} = \{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}\{\mathbf{U}_p\mathbf{\Lambda}_p\mathbf{V}_p^T\} = \mathbf{V}_p\mathbf{V}_p^T$$

$$\mathbf{N} = \mathbf{G}\mathbf{G}^{-g} = \{\mathbf{U}_p\mathbf{\Lambda}_p\mathbf{V}_p^T\}\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\} = \mathbf{U}_p\mathbf{U}_p^T$$

$$[\mathrm{cov}\,\mathbf{m}^{est}] = \mathbf{G}^{-g}[\mathrm{cov}\,\mathbf{d}]\mathbf{G}^{-gT} =$$

$$= \sigma_d^2\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}^T = \sigma_d^2\mathbf{V}_p\mathbf{\Lambda}_p^{-2}\mathbf{V}_p^T$$

large covariance if any λ_p are small

Is the Natural Solution the best solution?

Why restrict a priori information to the null space

when the data are known to be in error?

A solution that has slightly worse error but fits the a priori information better might be preferred ...