Lecture 12

Equality and Inequality Constraints

Syllabus

Lecture 01	Describing Inverse Problems
Lecture 02	Probability and Measurement Error, Part 1
Lecture 03	Probability and Measurement Error, Part 2
Lecture 04	The L_2 Norm and Simple Least Squares
Lecture 05	A Priori Information and Weighted Least Squared
Lecture 06	Resolution and Generalized Inverses
Lecture 07	Backus-Gilbert Inverse and the Trade Off of Resolution and Variance
Lecture 08	The Principle of Maximum Likelihood
Lecture 09	Inexact Theories
Lecture 10	Nonuniqueness and Localized Averages
Lecture 11	Vector Spaces and Singular Value Decomposition
Lecture 12	**Equality and Inequality Constraints**
Lecture 13	L_1 , L_∞ Norm Problems and Linear Programming
Lecture 14	Nonlinear Problems: Grid and Monte Carlo Searches
Lecture 15	Nonlinear Problems: Newton's Method
Lecture 16	Nonlinear Problems: Simulated Annealing and Bootstrap Confidence Intervals
Lecture 17	Factor Analysis
Lecture 18	Varimax Factors, Empirical Orthogonal Functions
Lecture 19	Backus-Gilbert Theory for Continuous Problems; Radon's Problem
Lecture 20	Linear Operators and Their Adjoints
Lecture 21	Fréchet Derivatives
Lecture 22	Exemplary Inverse Problems, incl. Filter Design
Lecture 23	Exemplary Inverse Problems, incl. Earthquake Location
Lecture 24	Exemplary Inverse Problems, incl. Vibrational Problems

Purpose of the Lecture

Review the Natural Solution and SVD

Apply SVD to other types of prior information
and to
equality constraints

Introduce Inequality Constraints and the
Notion of Feasibility

Develop Solution Methods

Solve Exemplary Problems

Part 1

Review the Natural Solution and SVD

subspaces

model parameters

$\mathbf{m}_p$ can affect data

$\mathbf{m}_0$ cannot affect data

data

$\mathbf{d}_p$ can be fit by model

$\mathbf{d}_0$ cannot be fit by any model

natural solution

determine $\mathbf{m}_p$ by solving $\mathbf{d}_p - \mathbf{G}\mathbf{m}_p = 0$

set $\mathbf{m}_0 = 0$

natural solution

determine $\mathbf{m}_p$ by solving $\mathbf{d}_p - \mathbf{G}\mathbf{m}_p = 0$

set $\mathbf{m}_0 = 0$

error reduced to its minimum $E = \mathbf{e}_0^T \mathbf{e}_0$

natural solution

determine $\mathbf{m}_p$ by solving $\mathbf{d}_p - \mathbf{G}\mathbf{m}_p = 0$

set $\mathbf{m}_0 = 0$

solution length reduced to its minimum $L = \mathbf{m}_p^T \mathbf{m}_p$

Singular Value Decomposition (SVD)

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{T}}$$

$N \times N$ matrix of eigenvectors

$$\mathbf{U} = [\mathbf{u}^{(1)} \quad \mathbf{u}^{(2)} \quad \mathbf{u}^{(3)} \quad \cdots \quad \mathbf{u}^{\mathrm{N}}]$$

$M \times M$ matrix of eigenvectors

$$\mathbf{V} = [\mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \mathbf{v}^{(3)} \quad \cdots \quad \mathbf{v}^{\mathrm{M}}]$$

$\boldsymbol{\Lambda}$ is an $N \times M$ diagonal matrix

singular value decomposition

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{T}}$$

$$\mathbf{U}^{\mathrm{T}}\mathbf{U}=\mathbf{I} \quad \text{and} \quad \mathbf{V}^{\mathrm{T}}\mathbf{V}=\mathbf{I}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \end{bmatrix}$$

suppose only p λ's are non-zero

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_{\mathrm{p}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

suppose only p λ's are non-zero

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{T}} \Rightarrow \mathbf{U}_{\mathrm{p}}\boldsymbol{\Lambda}_{\mathrm{p}}\mathbf{V}_{\mathrm{p}}^{\mathrm{T}}$$

only first *p* columns of **U**

only first *p* columns of **V**

$\mathbf{U}_p^T\mathbf{U}_p=\mathbf{I}$ and $\mathbf{V}_p^T\mathbf{V}_p=\mathbf{I}$
since vectors mutually pependicular
and of unit length

$\mathbf{U}_p\mathbf{U}_p^T\neq\mathbf{I}$ and $\mathbf{V}_p\mathbf{V}_p^T\neq\mathbf{I}$
since vectors do not span entire space

The Natural Solution

$$\mathbf{m}^{\text{est}} = \mathbf{V}_{\text{p}} \mathbf{\Lambda}_{\text{p}}^{-1} \mathbf{U}_{\text{p}}^{\text{T}} \mathbf{d}$$

The Natural Solution

$$\mathbf{m}^{\mathrm{est}} = \mathbf{V}_p \boldsymbol{\Lambda}_p^{-1} \mathbf{U}_p^T \mathbf{d}$$

natural generalized inverse $\mathbf{G}^{-g}$

resolution and covariance

$$\mathbf{R} = \mathbf{G}^{-g}\mathbf{G} = \{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}\{\mathbf{U}_p\mathbf{\Lambda}_p\mathbf{V}_p^T\} = \mathbf{V}_p\mathbf{V}_p^T$$

$$\mathbf{N} = \mathbf{G}\mathbf{G}^{-g} = \{\mathbf{U}_p\mathbf{\Lambda}_p\mathbf{V}_p^T\}\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\} = \mathbf{U}_p\mathbf{U}_p^T$$

$$[\mathrm{cov}\,\mathbf{m}^{est}] = \mathbf{G}^{-g}[\mathrm{cov}\,\mathbf{d}]\mathbf{G}^{-gT} =$$

$$= \sigma_d^2\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}\{\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\}^T = \sigma_d^2\mathbf{V}_p\mathbf{\Lambda}_p^{-2}\mathbf{V}_p^T$$

Part 2

Application of SVD to other types of prior information and to equality constraints

general solution to linear inverse problem

$$\mathbf{m}^{\text{gen}} = \mathbf{m}^{\text{par}} + \sum_{i=1}^{p} \alpha_i \mathbf{m}^{\text{null (i)}}$$

$$\mathbf{m} = \mathbf{V}_{\text{p}} \boldsymbol{\Lambda}_{\text{p}}^{-1} \mathbf{U}_{\text{p}}^{\text{T}} \mathbf{d} + \mathbf{V}_0 \boldsymbol{\alpha}$$

general minimum-error solution

$$\mathbf{m}^{\text{gen}} = \mathbf{m}^{\text{par}} + \sum_{i=1}^{p} \alpha_i \mathbf{m}^{\text{null (i)}}$$

2 lectures ago

$$\mathbf{m} = \mathbf{V}_{\text{p}} \boldsymbol{\Lambda}_{\text{p}}^{-1} \mathbf{U}_{\text{p}}^{\text{T}} \mathbf{d} + \mathbf{V}_0 \boldsymbol{\alpha}$$

general minimum-error solution

$$\mathbf{m}^{\text{gen}} = \mathbf{m}^{\text{par}} + \sum_{i=1}^{p} \alpha_i \mathbf{m}^{\text{null (i)}}$$

$$\mathbf{m} = \mathbf{V}_{\text{p}} \mathbf{\Lambda}_{\text{p}}^{-1} \mathbf{U}_{\text{p}}^{\text{T}} \mathbf{d} + \mathbf{V}_0 \boldsymbol{\alpha}$$

natural solution

plus amount **α** of null vectors

you can adjust $\boldsymbol{\alpha}$ to match whatever a priori information you want

$$\mathbf{m} = \mathbf{V}_p \boldsymbol{\Lambda}_p^{-1} \mathbf{U}_p^T \mathbf{d} + \mathbf{V}_0 \boldsymbol{\alpha}$$

for example

$\mathbf{m} = \langle \mathbf{m} \rangle$

by minimizing $L = ||\mathbf{m} - \langle \mathbf{m} \rangle||_2$ w.r.t. $\boldsymbol{\alpha}$

you can adjust $\boldsymbol{\alpha}$ to match whatever a priori information you want

$$\mathbf{m} = \mathbf{V}_p \boldsymbol{\Lambda}_p^{-1} \mathbf{U}_p^T \mathbf{d} + \mathbf{V}_0 \boldsymbol{\alpha}$$

for example

$\mathbf{m} = \langle\mathbf{m}\rangle$

by minimizing $L = ||\mathbf{m} - \langle\mathbf{m}\rangle||_2$ w.r.t. $\boldsymbol{\alpha}$

get $\boldsymbol{\alpha} = \mathbf{V}_0^T \langle\mathbf{m}\rangle$ so $\mathbf{m} = \mathbf{V}_p \boldsymbol{\Lambda}_p^{-1} \mathbf{U}_p^T \mathbf{d} + \mathbf{V}_0 \mathbf{V}_0^T \langle\mathbf{m}\rangle$

equality constraints

minimize E with constraint $\mathbf{Hm}=\mathbf{h}$

Step 1

find part of solution constrained by

$$\mathbf{Hm}=\mathbf{h}$$

SVD of **H** (not **G**)

$$\mathbf{H} = \mathbf{V_p}\boldsymbol{\Lambda}_\mathrm{p}\mathbf{U}_\mathrm{p}{}^\mathrm{T}$$

so

$$\mathbf{m}=\mathbf{V_p}\boldsymbol{\Lambda}_\mathrm{p}{}^{-1}\mathbf{U}_\mathrm{p}{}^\mathrm{T}\mathbf{h} + \mathbf{V}_0\boldsymbol{\alpha}$$

Step 2

convert $\mathbf{Gm=d}$

into and equation for $\boldsymbol{\alpha}$

$$\mathbf{G}\mathbf{V}_\mathbf{p}\boldsymbol{\Lambda}_p^{-1}\mathbf{U}_p^T\mathbf{h} + \mathbf{G}\mathbf{V}_0\boldsymbol{\alpha} = \mathbf{d}$$

and rearrange

$$[\mathbf{G}\mathbf{V}_0]\boldsymbol{\alpha} = [\mathbf{d} - \mathbf{G}\mathbf{V}_\mathbf{p}\boldsymbol{\Lambda}_p^{-1}\mathbf{U}_p^T\mathbf{h}]$$

$$\mathbf{G}'\boldsymbol{\alpha} = \mathbf{d}'$$

Step 3

solve $\mathbf{G}'\boldsymbol{\alpha} = \mathbf{d}'$

for $\boldsymbol{\alpha}$

using least squares

Step 4

reconstruct **m** from **α**

$$\mathbf{m} = \mathbf{V}_p \mathbf{\Lambda}_p^{-1} \mathbf{U}_p^T \mathbf{h} + \mathbf{V}_0 \boldsymbol{\alpha}$$

Part 3

Inequality Constraints and the Notion of Feasibility

Not all inequality constraints provide new information

$$x > 3$$
$$x > 2$$

Not all inequality constraints provide new information

$$x > 3$$

$$x > 2$$

follows from first constraint

Some inequality constraints are incompatible

$$x > 3$$
$$x < 2$$

Some inequality constraints are incompatible

$$x > 3$$
$$x < 2$$

nothing can be both bigger than 3 and smaller than 2 every row of the inequality constraint

$$\mathbf{Hm} \geq \mathbf{h}$$

divides the space of **m**
into two parts

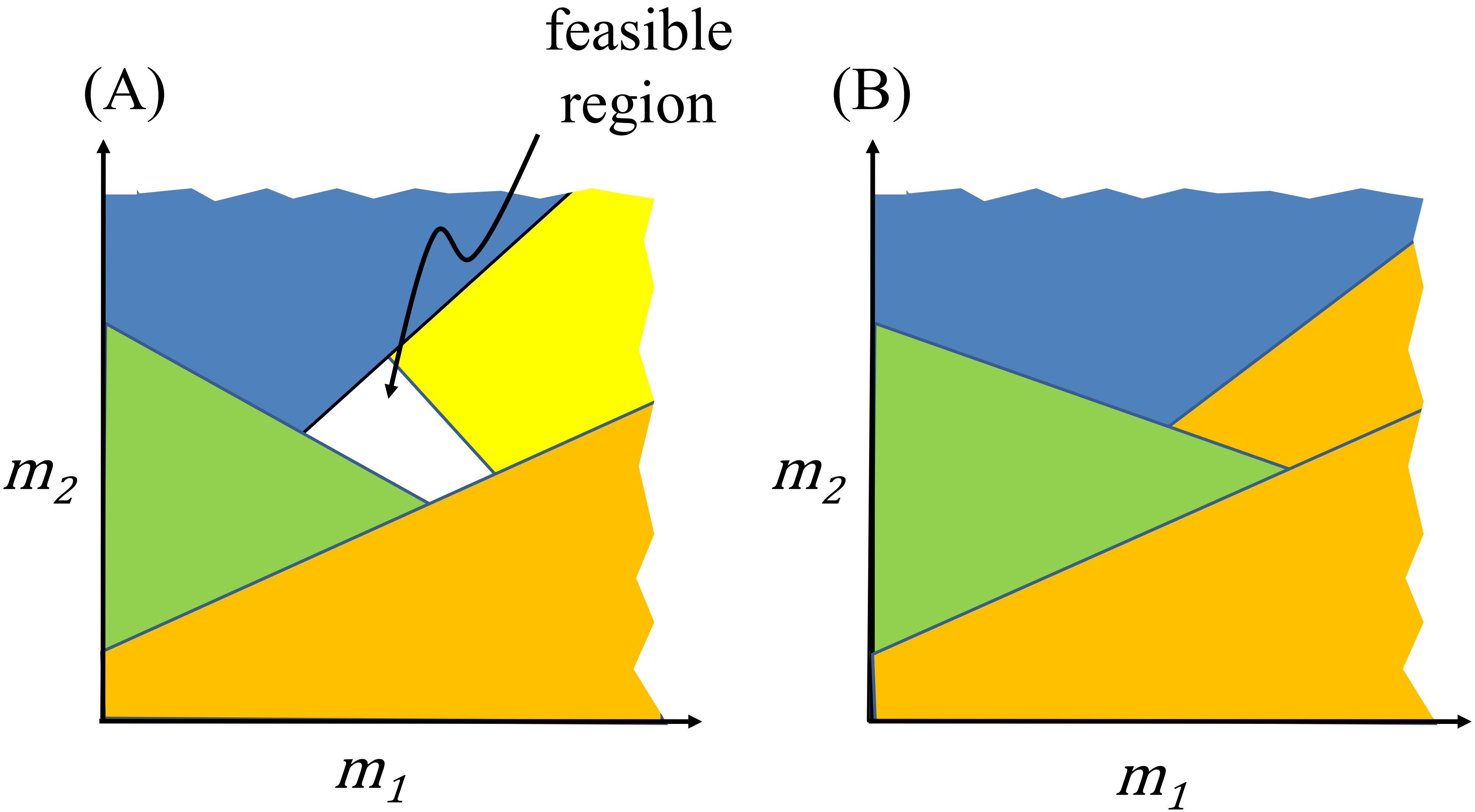
one where a solution is *feasible*
one where it is *infeasible*

the boundary is a planar surface

when all the constraints are considered together
they either create a feasible volume
or they don’t

if they do, then the solution must be in that volume

if they don’t, then no solution exists



(A)
feasible
region
m_2
m_1
(B)
m_2
m_1

now consider the problem of minimizing the error E

subject to inequality constraints $\mathbf{Hm} \geq \mathbf{h}$

if the global minimum is
inside the feasible region

then

the inequality constraints
have no effect on the solution

but
if the global minimum is
outside the feasible region

then

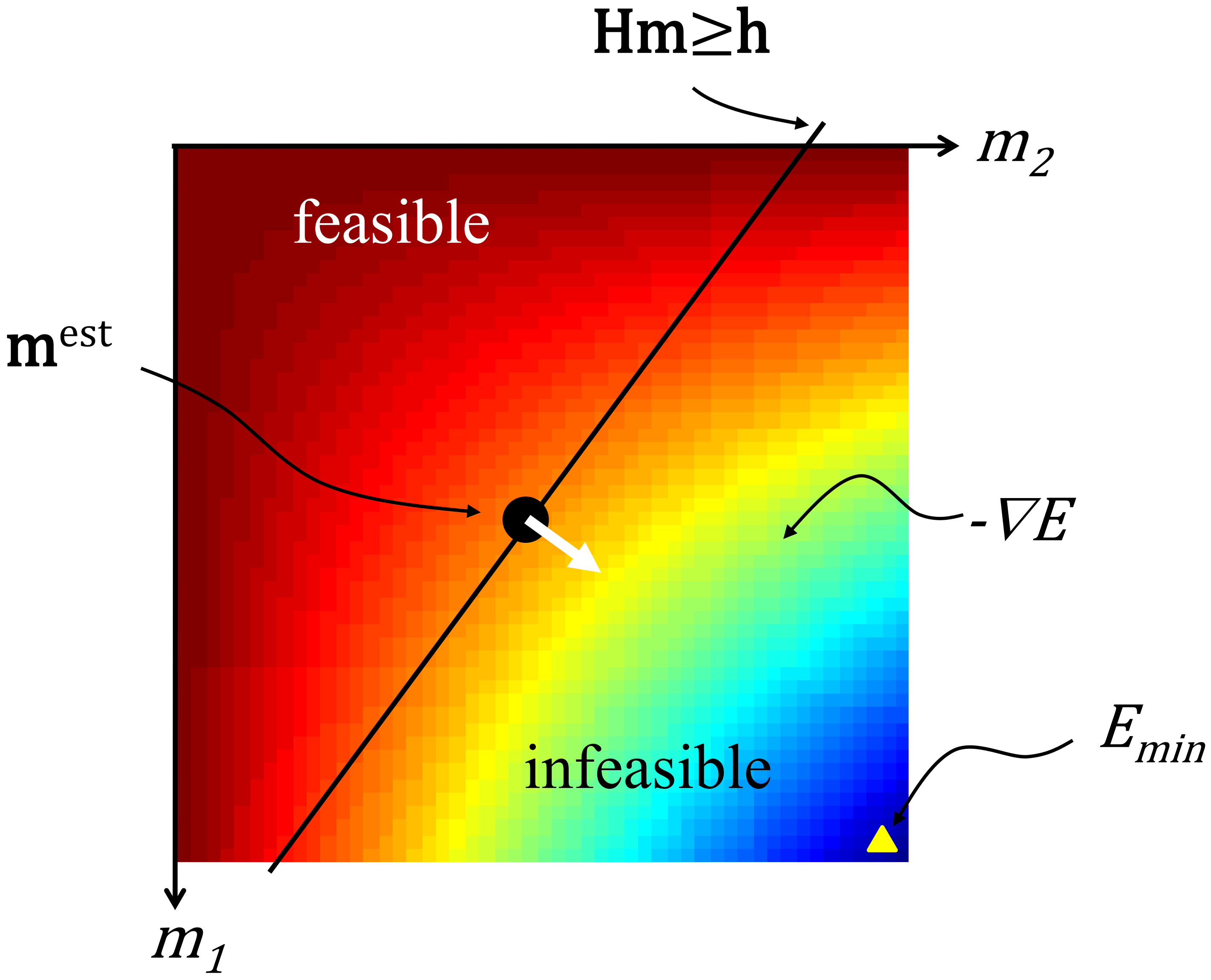
the solution is on the surface
of the feasible volume

but
if the global minimum is
outside the feasible region

then

the solution is on the surface
of the feasible volume

the point on the surface where E is the smallest

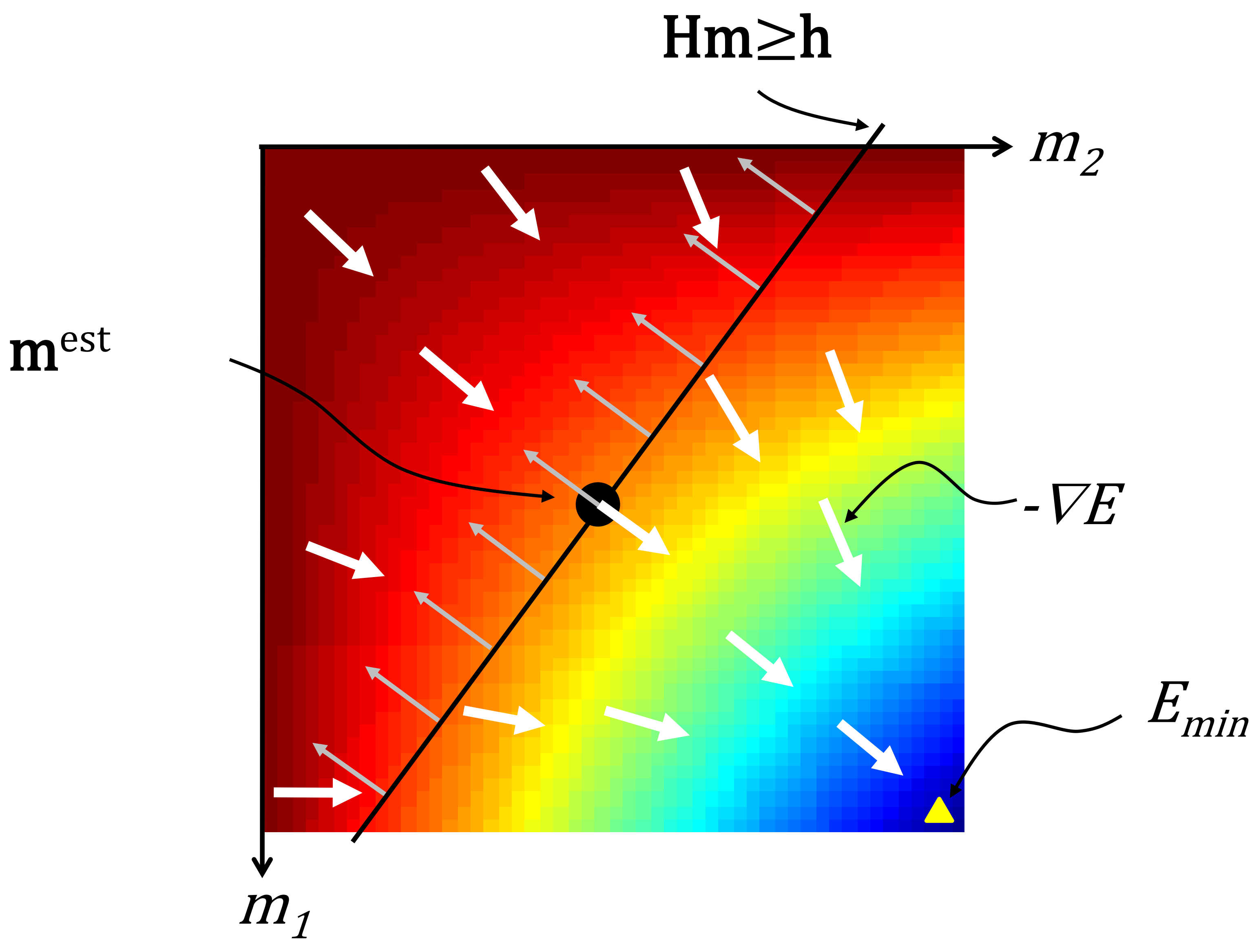


Hm≥h
m2
feasible
mest
-∇E
infeasible
Emin
m1

furthermore

the feasible-pointing normal to the surface
must be parallel to ∇E

else
you could slide the point along the surface
to reduce the error E



Hm≥h
m2
m^est
-∇E
E_min
m1

Kuhn – Tucker theorem

it's possible to find a vector **y** with $y_i \geq 0$ such that

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}] \cdot \mathbf{y} \quad \text{or} \quad -\mathbf{G}^{T}[\mathbf{d} - \mathbf{Gm}] = \mathbf{H}^{T}\mathbf{y}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector **y** with $\mathbf{y} \geq 0$ such that

feasible-pointing normals
to surface

$$\tfrac{1}{2}\nabla E = \overbrace{\nabla[\mathbf{Hm}]}^{\text{feasible-pointing normals to surface}} \cdot \mathbf{y} \quad \text{or} \quad -\mathbf{G}^T[\mathbf{d} - \mathbf{Gm}] = \mathbf{H}^T\mathbf{y}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector **y** with **y**≥0 such that

feasible-pointing normals
to surface

$$\tfrac{1}{2}\nabla E = \overbrace{\nabla[\mathbf{Hm}]}\cdot \mathbf{y} \quad \text{or} \quad -\mathbf{G}^T[\mathbf{d}-\mathbf{Gm}] = \mathbf{H}^T\mathbf{y}$$

the gradient
of the error

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector **y** with **y**≥0 such that

feasible-pointing normals to surface

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}] \cdot \mathbf{y} \quad \text{or} \quad -\mathbf{G}^T[\mathbf{d} - \mathbf{Gm}] = \mathbf{H}^T\mathbf{y}$$

the gradient of the error

is a non-negative combination of feasible normals

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector **y** with **y**≥0 such that

feasible-pointing normals to surface

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}]\cdot\mathbf{y} \quad \text{or} \quad -\mathbf{G}^T[\mathbf{d}-\mathbf{Gm}] = \mathbf{H}^T\mathbf{y}$$

y specifies the combination

the gradient of the error

is a non-negative combination of feasible normals

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector **y** with **y**≥0 such that

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}] \cdot \mathbf{y} \quad \text{or} \quad \mathbf{G}^T[\mathbf{d} - \mathbf{Gm}] = -\mathbf{H}^T\mathbf{y}$$

for linear case with
Gm=d

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector $\mathbf{y}$ with $\mathbf{y}\geq 0$ such that

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}] \cdot \mathbf{y} \quad \text{or} \quad \mathbf{G}^T[\mathbf{d} - \mathbf{Gm}] = -\mathbf{H}^T\mathbf{y}$$

some coefficients y_i are positive

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector **y** with **y**≥0 such that

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}] \cdot \mathbf{y} \quad \text{or} \quad \mathbf{G}^T[\mathbf{d} - \mathbf{Gm}] = -\mathbf{H}^T\mathbf{y}$$

some coefficients y_i are positive

the solution is on the corresponding constraint surface

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

it's possible to find a vector **y** with **y**≥0 such that

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}]\cdot\mathbf{y} \quad \text{or} \quad \mathbf{G}^T[\mathbf{d}-\mathbf{Gm}] = -\mathbf{H}^T\mathbf{y}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

some coefficients y_i
are zero

it's possible to find a vector $\mathbf{y}$ with $\mathbf{y} \geq 0$ such that

$$\tfrac{1}{2}\nabla E = \nabla[\mathbf{Hm}] \cdot \mathbf{y} \quad \text{or} \quad \mathbf{G}^T[\mathbf{d} - \mathbf{Gm}] = -\mathbf{H}^T\mathbf{y}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_E > 0 \\ \mathbf{y}_S = 0 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \mathbf{H}_E\mathbf{m} - \mathbf{h}_E = 0 \\ \mathbf{H}_S\mathbf{m} - \mathbf{h}_S > 0 \end{matrix}$$

some coefficients y_i are zero

the solution is on the feasible side of the corresponding constraint surface

Part 4

Solution Methods

simplest case

minimize E subject to $m_i > 0$
($\mathbf{H} = \mathbf{I}$ and $\mathbf{h} = 0$)

iterative algorithm with two nested loops

Step 1

Start with an initial guess for $\mathbf{m}$

The particular initial guess $\mathbf{m}=0$ is feasible

It has all its elements in $\mathbf{m}_E$
constraints satisfied in the equality sense

Step 2

Any model parameter m_i in $\mathbf{m}_E$ that has associated with it a negative gradient $[\nabla E]_i$ can be changed both to decrease the error and to remain feasible.

If there is no such model parameter in $\mathbf{m}_E$, the Kuhn – Tucker theorem indicates that this $\mathbf{m}$ is the solution to the problem.

Step 3

If some model parameter m_i in $\mathbf{m}_E$ has a corresponding negative gradient, then the solution can be changed to decrease the prediction error.

To change the solution, we select the model parameter corresponding to the most negative gradient and move it to the set $\mathbf{m}_S$.

All the model parameters in $\mathbf{m}_S$ are now recomputed by solving the system $\mathbf{G}_S\mathbf{m}'_S=\mathbf{d}_S$ in the least squares sense. The subscript S on the matrix indicates that only the columns multiplying the model parameters in $\mathbf{m}_S$ have been included in the calculation.

All the $\mathbf{m}_E$'s are still zero. If the new model parameters are all feasible, then we set $\mathbf{m} = \mathbf{m}'$ and return to Step 2.

Step 4

If some of the elements of $\mathbf{m'}_S$ are infeasible, however, we cannot use this vector as a new guess for the solution.

So, we compute the change in the solution and add as much of this vector as possible to the solution $\mathbf{m}_S$ without causing the solution to become infeasible.

We therefore replace $\mathbf{m}_S$ with the new guess $\mathbf{m}_S + \alpha\,\delta\mathbf{m}$, where is the largest choice that can be made without some $\mathbf{m}_S$ becoming infeasible. At least one of the m_{Si}'s has its constraint satisfied in the equality sense and must be moved back to $\mathbf{m}_E$. The process then returns to Step 3.

In MatLab

```
mest = lsqnonneg(G,dobs);
```

example

gravitational field depends upon density

via the inverse square law

example

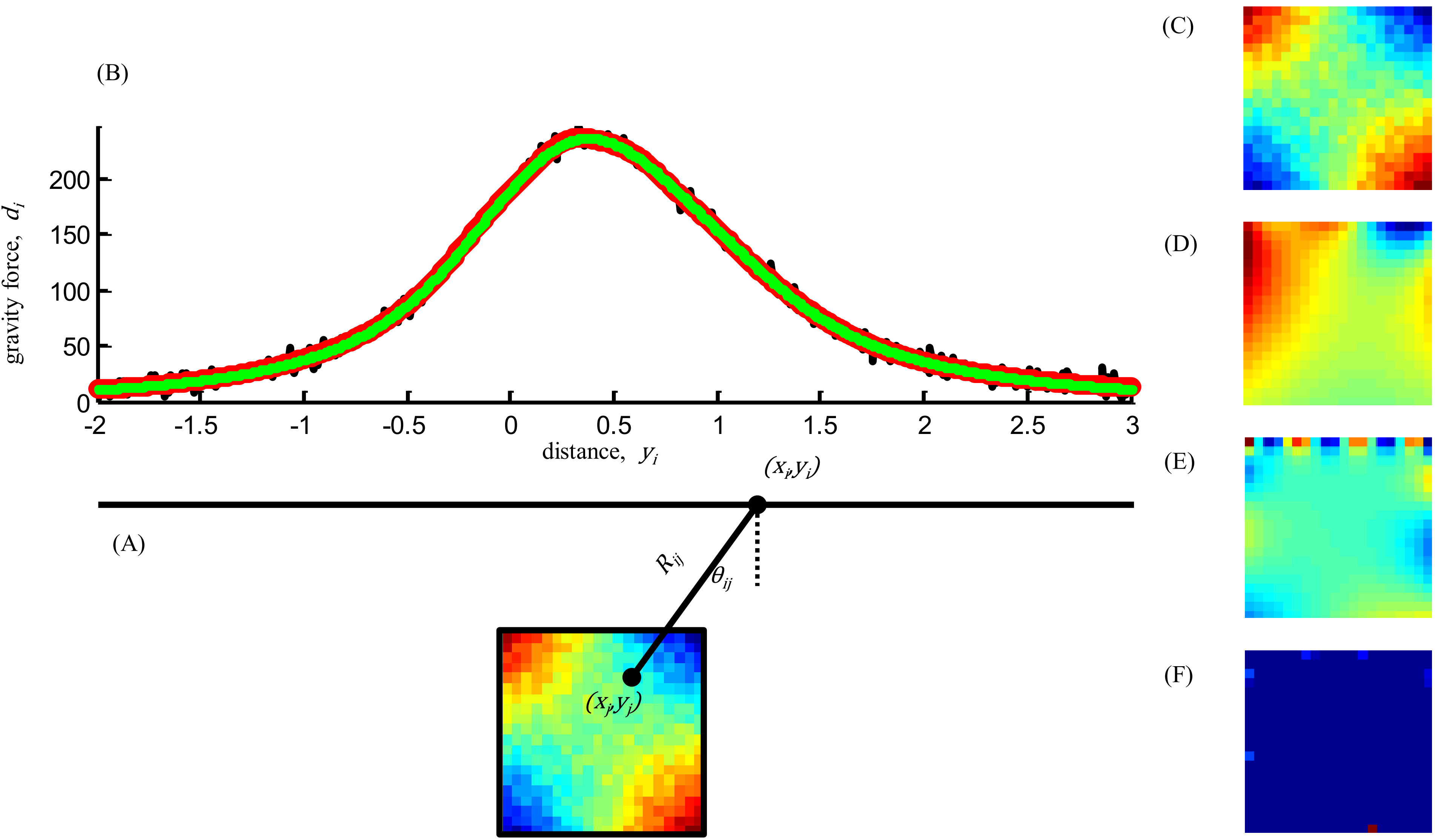
gravitational force depends upon density

observations

model parameters

via the inverse square law

theory



(B)
gravity force, d_i
0
50
100
150
200
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
distance, y_i
(x_i, y_i)
(A)
R_{ij}
θ_{ij}
(x_j, y_j)
(C)
(D)
(E)
(F)

more complicated case

minimize $||\mathbf{m}||_2$ subject to $\mathbf{Hm} \geq \mathbf{h}$

this problem is solved by transformation to the previous problem solve by non-negative least squares

$$\mathbf{G}'\mathbf{m}' = \mathbf{d}' = \begin{bmatrix} \mathbf{H}^{\mathrm{T}} \\ \mathbf{h}^{\mathrm{T}} \end{bmatrix} \mathbf{m}' = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

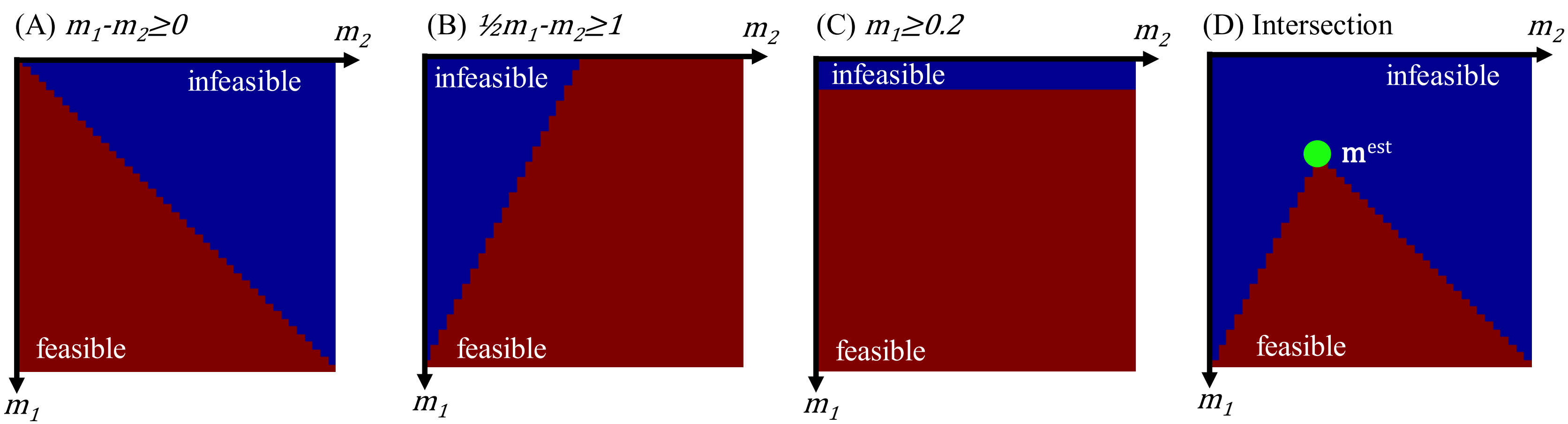
then compute m_i as

$$-e'_i / e'_{M+1}$$

with $\mathbf{e}' = \mathbf{d}' - \mathbf{G}'\mathbf{m}'$

In MatLab

```
Gp = [H, h]';
dp = [zeros(1,length(H(1,:))), 1]';
mp = lsqnonneg(Gp,dp);
ep = dp - Gp*mp;
m = -ep(1:end-1)/ep(end);
```


(A) m_1-m_2≥0
m_2
infeasible
feasible
m_1
(B) ½m_1-m_2≥1
m_2
infeasible
feasible
m_1
(C) m_1≥0.2
m_2
infeasible
feasible
m_1
(D) Intersection
m_2
infeasible
$\mathbf{m}^{est}$
feasible
m_1

yet more complicated case

minimize $||\mathbf{d}-\mathbf{Gm}||_2$ subject to $\mathbf{Hm} \geq \mathbf{h}$

this problem is solved by transformation to the previous problem

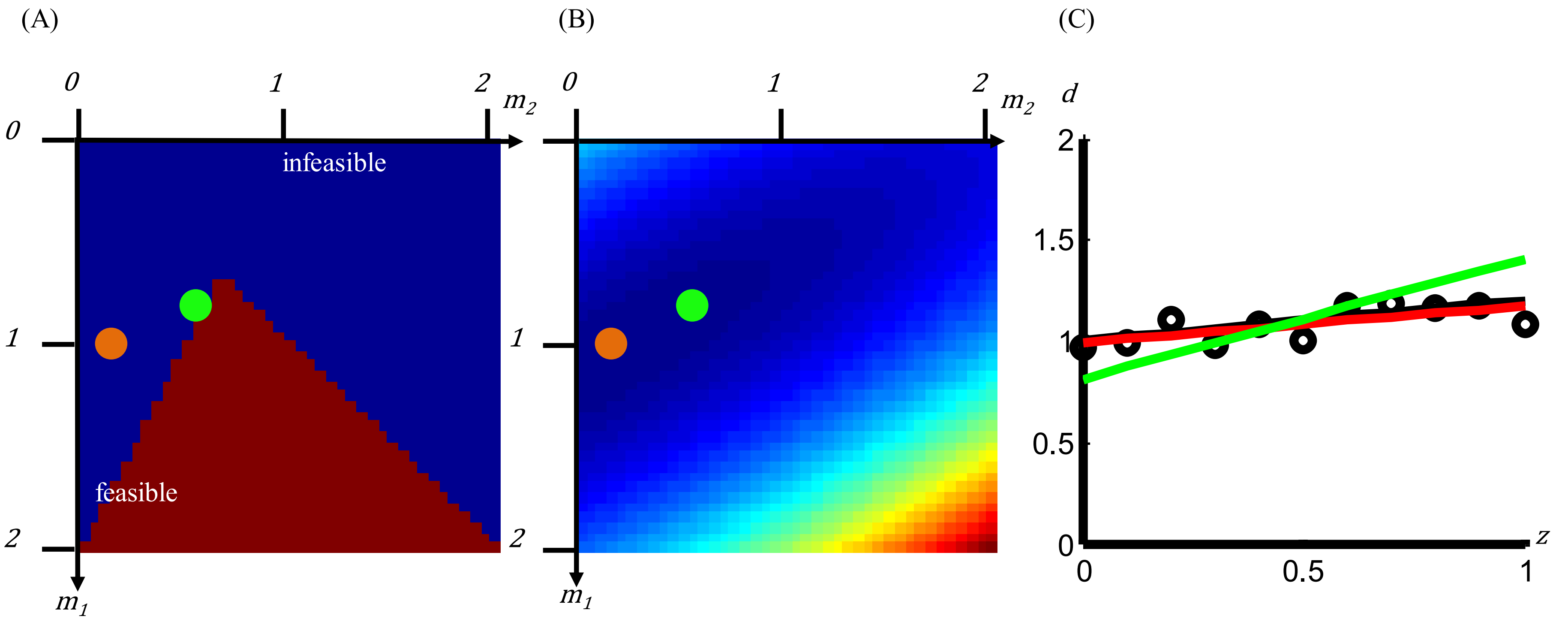
$$\text{minimize } \|\mathbf{m}'\| \quad \text{subject to } \mathbf{H}'\mathbf{m}' \geq \mathbf{h}'$$

where

$$\{-\mathbf{H}\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\}\,\mathbf{m}' \geq \{\mathbf{h} - \mathbf{H}\mathbf{V}_p\mathbf{\Lambda}_p^{-1}\mathbf{U}_p^T\mathbf{d}\} \quad \text{or} \quad \mathbf{H}'\mathbf{m}' \geq \mathbf{h}'$$

and

$$\mathbf{m} = \mathbf{V}_p\mathbf{\Lambda}_p^{-1}[\mathbf{d}_p - \mathbf{m}'] = \mathbf{V}_p\mathbf{\Lambda}_p^{-1}[\mathbf{U}_p^T\mathbf{d} - \mathbf{m}']$$


(A)
0
1
2
m_2
0
1
2
m_1
infeasible
feasible
(B)
0
1
2
m_2
1
2
m_1
(C)
d
2
1.5
1
0.5
0
0
0.5
1
z

In MatLab

```
[Up, Lp, Vp] = svd(G,0);
lambda = diag(Lp);
rlambda = 1./lambda;
Lpi = diag(rlambda);

% transformation 1
Hp = -H*Vp*Lpi;
hp = h + Hp*Up'*dobs;

% transformation 2
Gp = [Hp, hp]';
dp = [zeros(1,length(Hp(1,:))), 1]';
mpp = lsqnonneg(Gp,dp);
ep = dp - Gp*mpp;
mp = -ep(1:end-1)/ep(end);

% take mp back to m
mest = Vp*Lpi*(Up'*dobs-mp);
dpre = G*mest;
```