Lecture 19

Continuous Problems: Backus-Gilbert Theory and Radon’s Problem

Syllabus

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Lecture 20	Linear Operators and Their Adjoints
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Lecture 23	Exemplary Inverse Problems, incl. Earthquake Location
Lecture 24	Exemplary Inverse Problems, incl. Vibrational Problems

Purpose of the Lecture

Extend Backus-Gilbert theory to continuous problems

Discuss the conversion of continuous inverse problems to discrete problems

Solve Radon’s Problem
the simplest tomography problem

Part 1

Backus-Gilbert Theory

Continuous Inverse Theory

the data are discrete
but
the model parameter is a continuous function

One or several dimensions

$$d_i = \int_a^b G_i(z)\, m(z)\, \mathrm{dz}$$

$$d_i = \int_V G_i(\mathbf{x})\, m(\mathbf{x})\, \mathrm{d}^{\mathrm{L}} x$$

One or several dimensions

$$d_i = \int_a^b G_i(z)\, m(z)\, \mathrm{d}z$$

data model function

$$d_i = \int_V G_i(\mathbf{x})\, m(\mathbf{x})\, \mathrm{d}^L x$$

hopeless to try to determine estimates of model function at a particular depth

$$m(z_0) = ?$$

localized average is the only way to go

$$m^{avg}(\mathbf{x}') = \int R(\mathbf{x}',\mathbf{x})\, m^{true}(\mathbf{x})\ \mathrm{d}^{\mathrm{L}}x$$

hopeless to try to determine estimates of model function at a particular depth

$$m(z_0) = ?$$

the problem is that an integral, such as the data kernel integral, does not depend upon the value of m(z) at a "single point" z_0

localized average is the only way to go

$$m^{avg}(\mathbf{x}') = \int R(\mathbf{x}', \mathbf{x})\, m^{true}(\mathbf{x})\; \mathrm{d}^{L} x$$

continuous version of resolution matrix

let's retain the idea that the "solution" depends *linearly* on the data

$$m^{avg}(\mathbf{x}') = \sum_{i=1}^{N} G_i^{-g}(\mathbf{x}') d_i :$$

let's retain the idea that the "solution"
depends *linearly* on the data

$$m^{avg}(\mathbf{x}') = \sum_{i=1}^{N} G_i^{-g}(\mathbf{x}')d_i :$$

continuous version of generalized inverse

implies a formula for *R*

$$m^{avg}(\mathbf{x}') = \sum_{i=1}^{N} G_i^{-g}(\mathbf{x}')\, d_i \,:$$

$$d_i = \int_V G_i(\mathbf{x})\, m(\mathbf{x})\, \mathrm{d}^{\mathrm{L}} x$$

$$m^{avg}(\mathbf{x}') = \int R(\mathbf{x}', \mathbf{x})\, m^{true}(\mathbf{x})$$

$$R(\mathbf{x}', \mathbf{x}) = \sum_{i=1}^{N} G_i^{-g}(\mathbf{x}')\, G_i(\mathbf{x})$$

comparison to discrete case

$\langle \mathbf{m} \rangle = \mathbf{G}^{-g}\mathbf{d}$

$\mathbf{d} = \mathbf{G}\mathbf{m}$

$\langle \mathbf{m} \rangle = \mathbf{R}\mathbf{m}$

$\mathbf{R} = \mathbf{G}^{-g}\mathbf{G}$

implies a formula for *R*

$$m^{avg}(\mathbf{x}') = \sum_{i=1}^{N} G_i^{-g}(\mathbf{x}') d_i \;:$$

$$d_i = \int_V G_i(\mathbf{x})\, m(\mathbf{x})\, \mathrm{d}^{\mathrm{L}} x$$

$$m^{avg}(\mathbf{x}') = \int R(\mathbf{x}', \mathbf{x})\, m^{true}(\mathbf{x})$$

$$R(\mathbf{x}', \mathbf{x}) = \sum_{i=1}^{N} G_i^{-g}(\mathbf{x}') G_i(\mathbf{x})$$

Now define the spread of resolution as

$$J(\mathbf{x}') = \int w(\mathbf{x}',\mathbf{x})\, R^2(\mathbf{x}',\mathbf{x})\ \mathrm{d}^{\mathrm{L}}x$$

fine generalized inverse

that minimizes the spread J
with the constraint that

$$\int R\ (\mathbf{x}', \mathbf{x})\ \mathrm{d}^{L}x = 1$$

$$J(\mathbf{x}') = \int w(\mathbf{x}',\mathbf{x})\, R(\mathbf{x}',\mathbf{x})\, R(\mathbf{x}',\mathbf{x})\ \mathrm{d}^{\mathrm{L}}x$$

$$= \int w(\mathbf{x}',\mathbf{x}) \sum_{i=1}^{N} G_i^{-g}(\mathbf{x}')G_i(\mathbf{x}) \sum_{j=1}^{N} G_j^{-g}(\mathbf{x}')G_j(\mathbf{x})\ \mathrm{d}^{\mathrm{L}}x$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} G_i^{-g}(\mathbf{x}')G_j^{-g}(\mathbf{x}') \int w(\mathbf{x}',\mathbf{x})G_i(\mathbf{x})G_j(\mathbf{x})\mathrm{d}^{\mathrm{L}}x$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} G_i^{-g}(\mathbf{x}')G_j^{-g}(\mathbf{x}')\,[\mathbf{S}(\mathbf{x}')]_{ij}$$

$$[\mathbf{S}(\mathbf{x}')]_{ij} = \int w(\mathbf{x}',\mathbf{x})G_i(\mathbf{x})G_j(\mathbf{x})\mathrm{d}^{\mathrm{L}}x$$

J has exactly the same form as the discrete case

only the definition of **S** is different

Hence the solution is the same as in the discrete case

$$G_l^{-g}(\mathbf{x}') = \frac{\sum_{i=1}^{N} u_i [\mathbf{S}^{-1}(\mathbf{x}')]_{il}}{\sum_{i=1}^{N} \sum_{j=1}^{N} u_i [\mathbf{S}^{-1}(\mathbf{x}')]_{ij} u_j}$$

where

$$[\mathbf{S}(\mathbf{x}')]_{ij} = \int w(\mathbf{x}', \mathbf{x}) G_i(\mathbf{x}) G_j(\mathbf{x}) \mathrm{d}^L x$$

$$u_i = \int G_i(\mathbf{x}) \, \mathrm{d}^L x$$

furthermore, just as we did in the discrete case, we can add the size of the covariance

$$\text{minimize } J'(\mathbf{x}') = \alpha \int w(\mathbf{x}', \mathbf{x})\, R^2(\mathbf{x}', \mathbf{x})\ \mathrm{d}^L x + \ (1-\alpha)\ \mathrm{var}[m^{avg}(\mathbf{x}')\,]$$

where

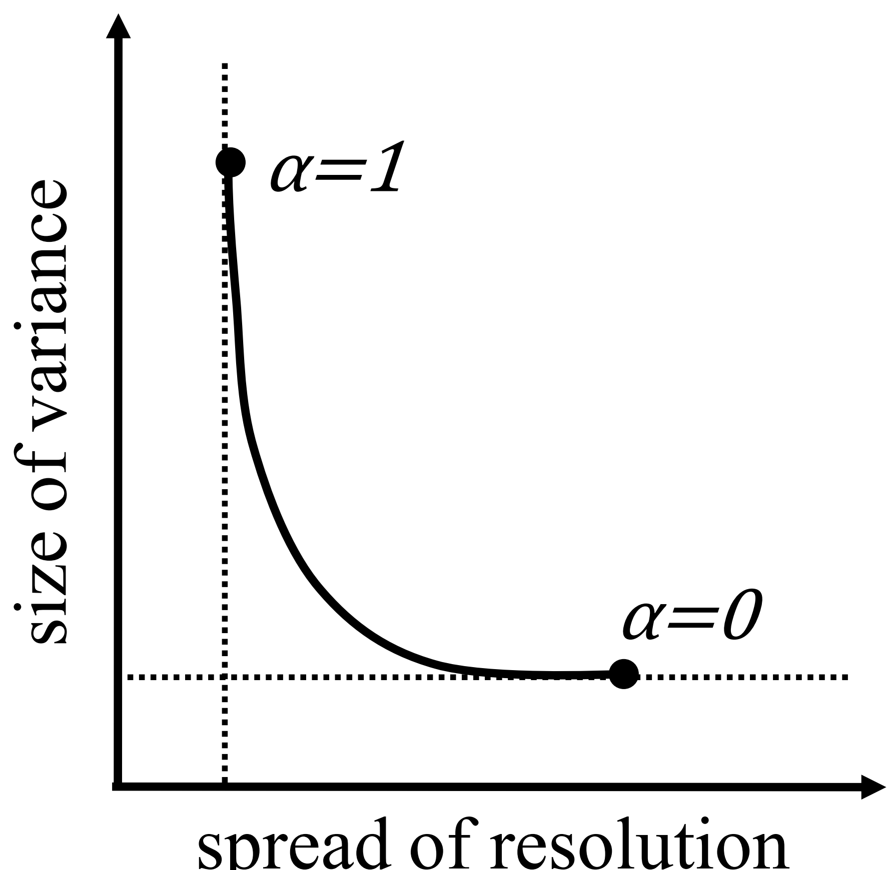
$$\mathbf{var}[m^{avg}(\mathbf{x}')\,] = \sum_{i=1}^{N} \sum_{j=1}^{N} G_i^{-g}(\mathbf{x}')\ [\mathrm{cov}\ \mathbf{d}]_{ij}\ G_j^{-g}(\mathbf{x}')$$

as before

this just changes the definition of **S**

$$[\mathbf{S}'(\mathbf{x}')]_{ij} = \alpha \int w(\mathbf{x}', \mathbf{x}) G_i(\mathbf{x}) G_j(\mathbf{x}) \mathrm{d}^L x + (1-\alpha)[\mathrm{cov}\,\mathbf{d}]_{ij}$$

and leads to a trade-off of resolution and variance



Part 2

Approximating a Continuous Problem as a Discrete Problem

approximation using finite number of known functions

$$m(\mathbf{x}) \approx \sum_{j=1}^{M} m_j f_j(\mathbf{x})$$

approximation using finite number of known functions

$$m(\mathbf{x}) \approx \sum_{j=1}^{M} m_j f_j(\mathbf{x})$$

continuous function

known functions

unknown coefficients
= discrete model parameters

posssible $f_j(\mathbf{x})$'s

voxels (and their lower dimension equivalents)

polynomials

splines

Fourier (and similar) series

and many others

does the choice of $f_j(\mathbf{x})$ matter?

Yes!

The choice implements prior information about the properties of the solution

The solution will be different depending upon the choice

conversion to discrete **Gm=d**

$$d_i = \int_V G_i(\mathbf{x})\, m(\mathbf{x})\, \mathrm{d}^{\mathrm{L}}x$$

$$d_i = \int G_i(\mathbf{x}) \sum_{j=1}^{M} m_j f_j(\mathbf{x})\, \mathrm{d}^{\mathrm{L}}x = \sum_{j=1}^{M} \left\{ \int G_i(\mathbf{x}) f_j(\mathbf{x})\, \mathrm{d}^{\mathrm{L}}x \right\} m_j$$

$$d_i = \sum_{j=1}^{M} G_{ij} m_j \quad \text{with} \quad G_{ij} = \int G_i(\mathbf{x}) f_j(\mathbf{x})\, \mathrm{d}^{\mathrm{L}}x$$

special case of voxels

$$f_i(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ inside } V_i \\ 0 & \text{otherwise} \end{cases}$$

size controlled by the scale of variation of $m(\mathbf{x})$

$$G_{ij} = \int G_i(\mathbf{x}) f_j(\mathbf{x})\, \mathrm{d}^L x \quad \Rightarrow \quad G_{ij} = \int_{V_j} G_i(\mathbf{x})\, \mathrm{d}^L x$$

integral over voxel j

approximation when $G_i(\mathbf{x})$ slowly varying

$$G_{ij} = \int_{V_j} G_i(\mathbf{x}) \, \mathrm{d}^{\mathrm{L}} x \Rightarrow \approx G_i\left(\mathbf{x}^{(\mathrm{j})}\right) V_j$$

center of voxel j

size controlled by the scale of variation of $G_i(\mathbf{x})$

more stringent condition than
scale of variation of $m(\mathbf{x})$

Part 3

Tomography

Greek Root

tomos

a cut, cutting, slice, section

“tomography”
as it is used in geophysics

data are line integrals of the model function

$$d_i = \int_{\underbrace{C_i}_{\text{curve } i}} m[x(s), y(s)]\, \mathrm{d}s$$

you can force this into the form

$$d_i = \int_V G_i(\mathbf{x})\, m(\mathbf{x})\, \mathrm{d}^{\mathrm{L}} x$$

if you want

$$d_i = \iint \underbrace{m(x, y)\delta\{x(s) - x_i[y(s)]\}\frac{\mathrm{d}s}{\mathrm{d}y}}_{G_i(\mathbf{x})} \mathrm{d}x\, dy = \int_{C_i} m[x(s), y(s)]\mathrm{d}s$$

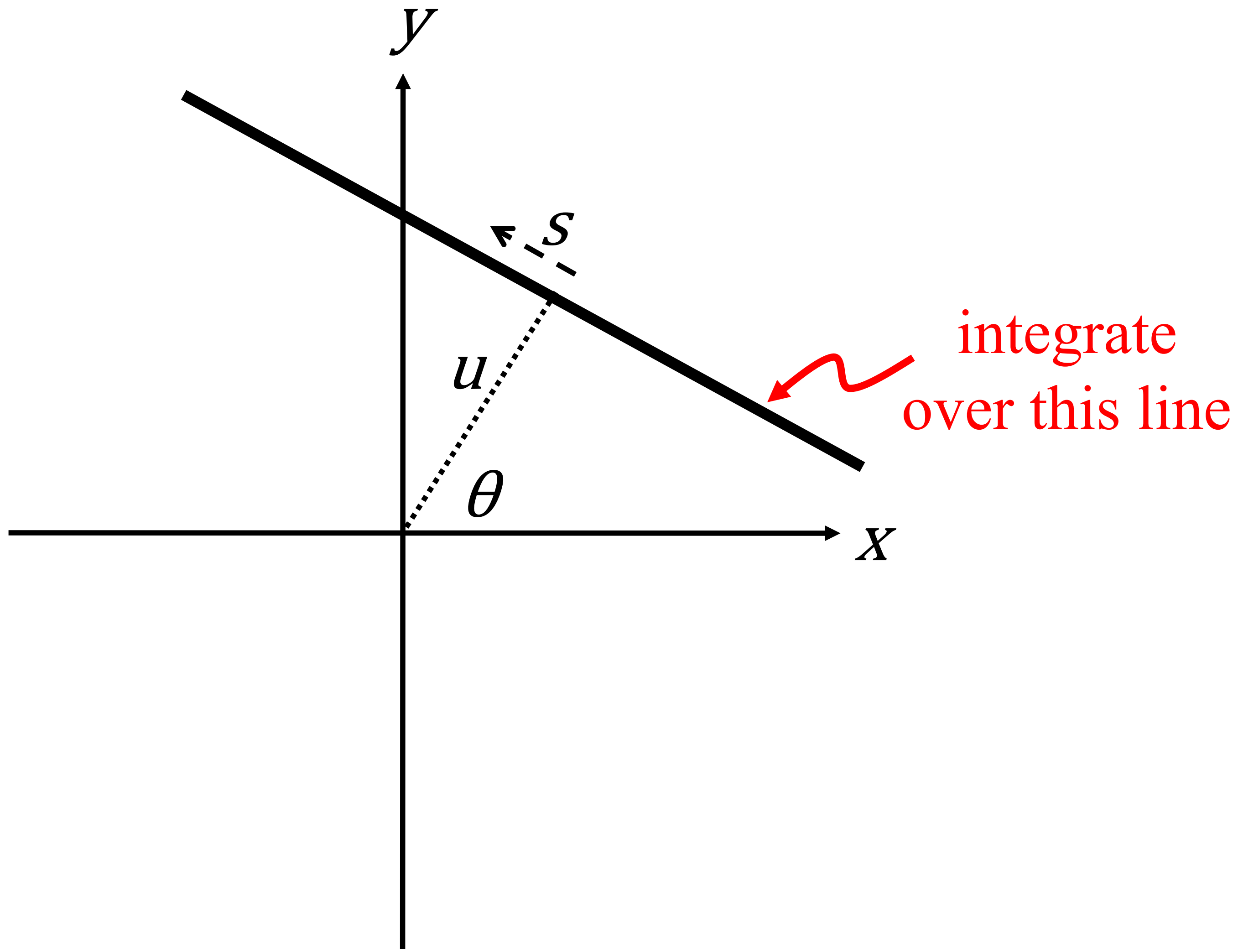
but the Dirac delta function is not square-integrable, which leads to problems

Radon's Problem

straight line rays

data d treated as a continuous variable

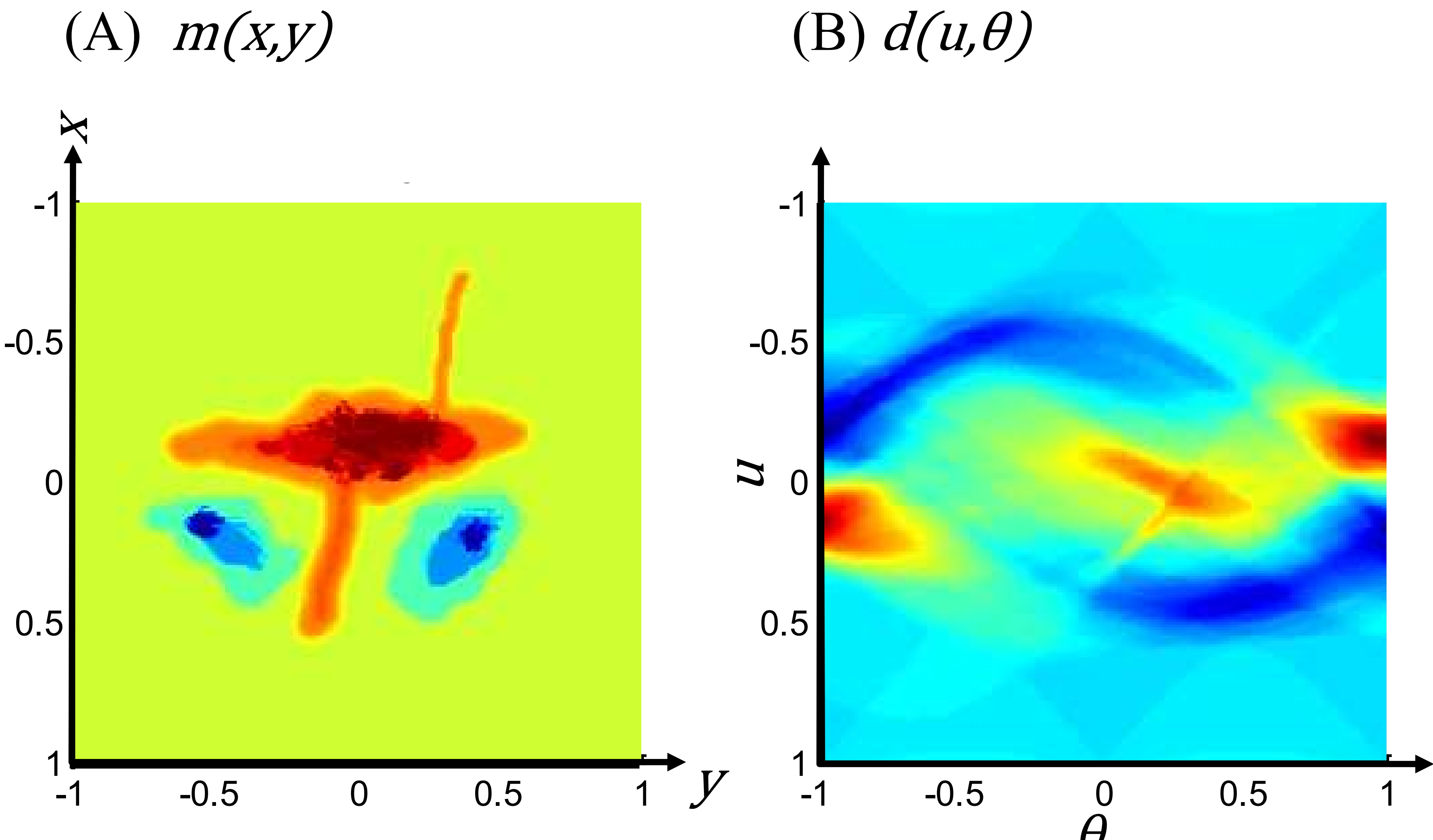
(u,θ) coordinate system for Radon Transform



Radon Transform

$$m(x,y) \rightarrow d(u,\theta)$$

$$d(u,\theta) = \int_{-\infty}^{+\infty} m(x = u\cos\theta - s\sin\theta\,, y = u\sin\theta + s\cos\theta)\,\mathrm{d}s$$


(A) m(x,y)
(B) d(u,θ)
x
-1
-0.5
0
0.5
1
y
-1
-0.5
0
0.5
1
u
θ

Inverse Problem

find $m(x,y)$ given $d(u,\theta)$

Solution via Fourier Transforms

$$\hat{f}(k_x) = \int_{-\infty}^{+\infty} f(x)\exp(ik_x x)\,\mathrm{d}x \quad \text{and} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k_x)\exp(-ik_x x)\,\mathrm{d}k_x$$

$x \rightarrow k_x$ $\qquad\qquad$ $k_x \rightarrow x$

$$d(u,\theta) = \int_{-\infty}^{+\infty} m(x = u\cos\theta - s\sin\theta\,, y = u\sin\theta + s\cos\theta)\,\mathrm{d}s$$

now Fourier transform
$u \rightarrow k_u$

$$\hat{d}(k_u,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(u\cos\theta - s\sin\theta\,, u\sin\theta + s\cos\theta)\,\mathrm{d}s\,\exp(ik_u u)\,\mathrm{d}u$$

now change variables
$(u,\theta) \rightarrow (x,y)$

$$\hat{d}(k_u,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(x,y)\;\exp(ik_u\cos\theta\, x + ik_u\sin\theta\, y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \hat{\hat{m}}\left(k_x = k_u\cos\theta\,, k_y = k_u\sin\theta\right)$$

$$d(u,\theta) = \int_{-\infty}^{+\infty} m(x = u\cos\theta - s\sin\theta\,, y = u\sin\theta + s\cos\theta)\,\mathrm{ds}$$

now Fourier transform

$u \to k_u$

$$\hat{d}(k_u,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(u\cos\theta - s\sin\theta\,, u\sin\theta + s\cos\theta)\,\mathrm{ds}\,\exp(ik_u u)\,\mathrm{d}u$$

now change variables

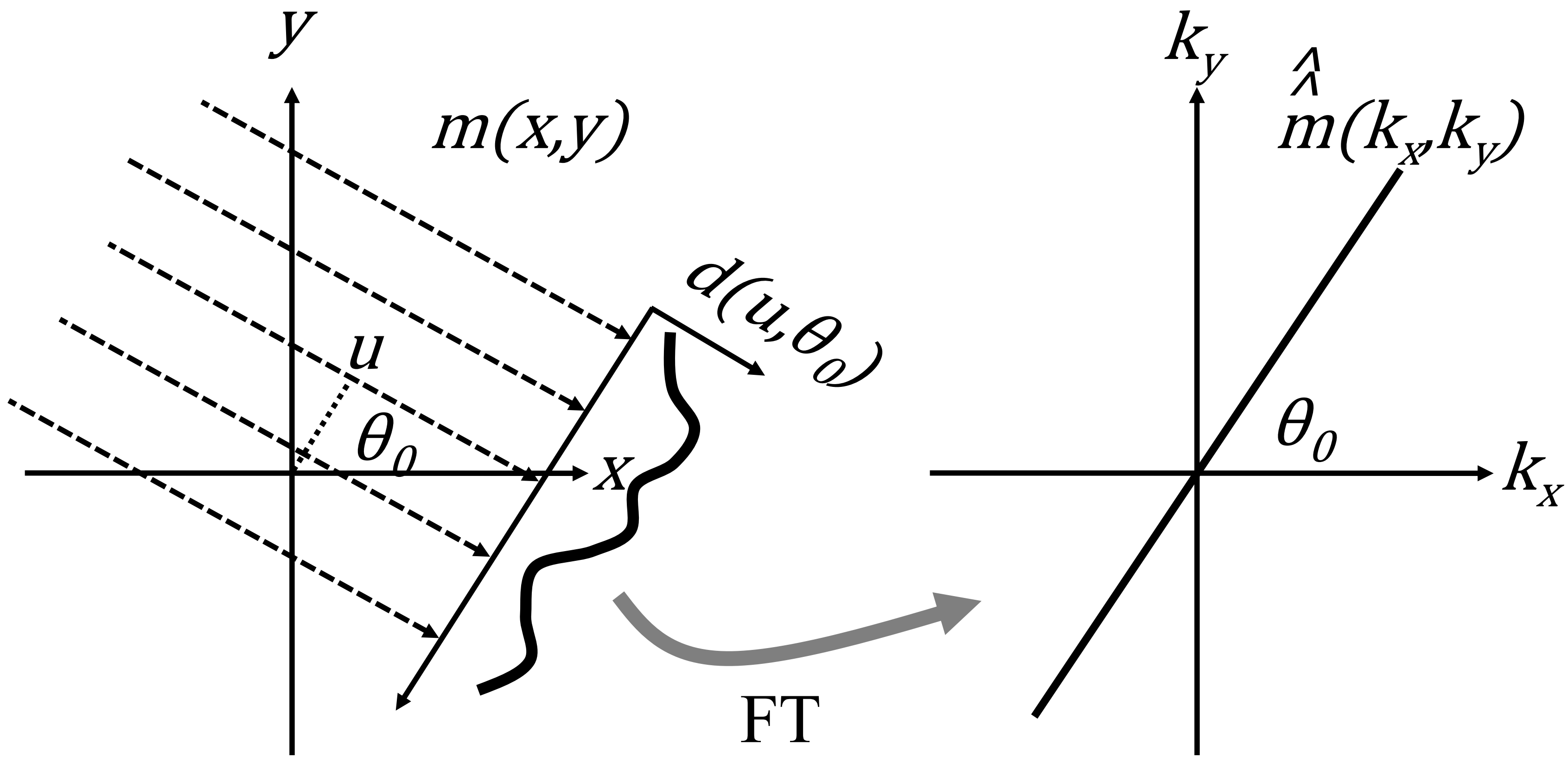
$(s,u) \to (x,y)$ *J=1, by the way*

$$\hat{d}(k_u,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m(x,y)\;\exp(ik_u\cos\theta\,x + ik_u\sin\theta\,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \hat{\hat{m}}\left(k_x = k_u\cos\theta\,, k_y = k_u\sin\theta\right)$$

Fourier transform of *d(u,θ)*

Fourier transform of *m(x,y)* evaluated on a line of slope θ

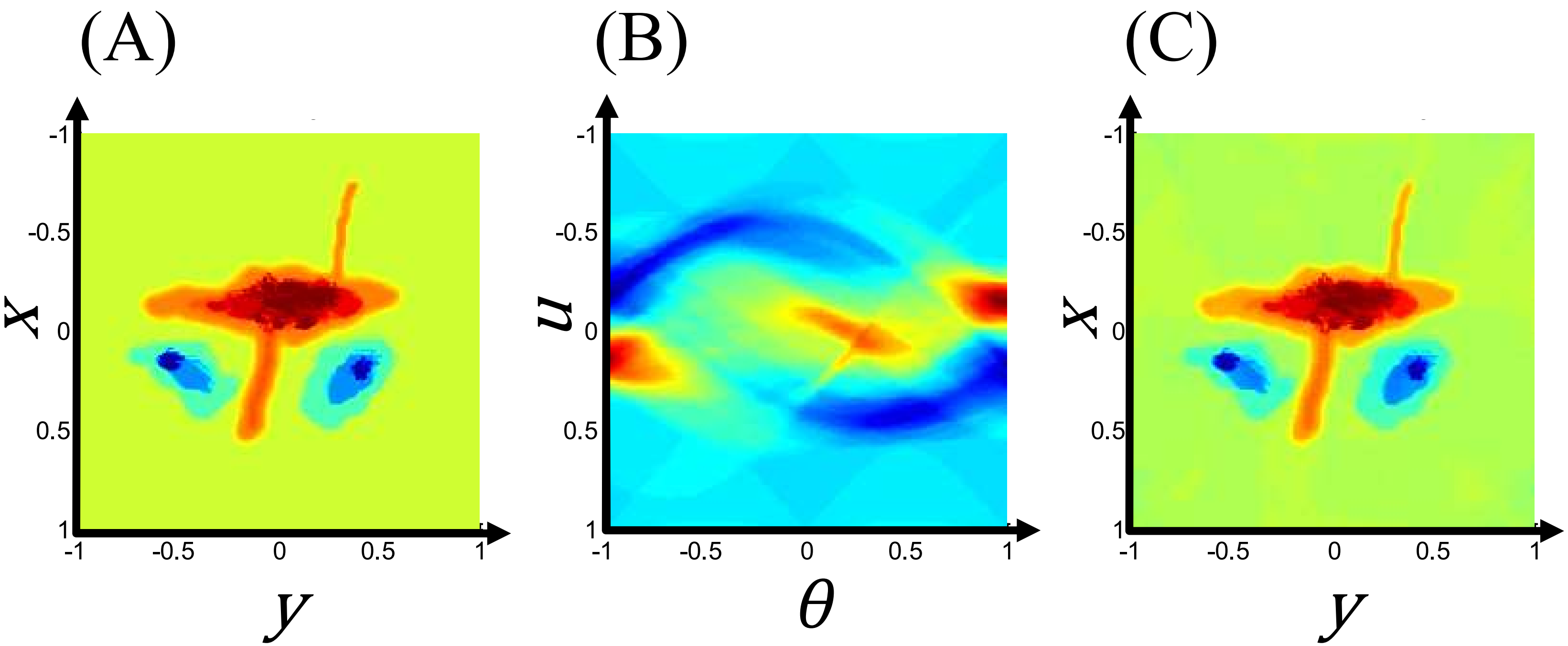


y
m(x,y)
u
θ_0
x
$d(u,\theta_0)$
FT
k_y
$\hat{m}(k_x,k_y)$
θ_0
k_x

Learned two things

1. Proof that solution exists and unique, based on “well-known” properties of Fourier Transform

2. Recipe how to invert a Radon transform using Fourier transforms


(A)
(B)
(C)
x
u
x
y
θ
y
-1
-0.5
0
0.5
1