Lecture 22

Exemplary Inverse Problems including Filter Design

Syllabus

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Lecture 15	Nonlinear Problems: Newton's Method
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Lecture 23	Exemplary Inverse Problems, incl. Earthquake Location
Lecture 24	Exemplary Inverse Problems, incl. Vibrational Problems

Purpose of the Lecture

solve a few exemplary inverse problems

image deblurring
deconvolution filters
minimization of cross-over errors

Part 1

image deblurring

three point blur
(applied to each row of pixels)

$$\mathbf{G} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

null vectors are highly oscillatory

$$\mathbf{m}^{(1)\text{null}} = [1 \quad 0 \quad -1 \quad 1 \quad 0 \quad -1 \quad \cdots \quad 1 \quad 0 \quad -1]^{\mathrm{T}}$$

$$\mathbf{m}^{(2)\text{null}} = [0 \quad 1 \quad -1 \quad 0 \quad 1 \quad -1 \quad \cdots \quad 0 \quad 1 \quad -1]^{\mathrm{T}}$$

solve with minimum length

$$\mathbf{m}^{\text{est}} = \mathbf{G}^{\text{T}}[\mathbf{G}\mathbf{G}^{\text{T}}]^{-1}\mathbf{d}^{\text{obs}}$$

note that $\mathbf{GG}^{\mathrm{T}}$ can deduced analytically

$$\mathbf{GG}^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 3 & 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

and is Toeplitz

might lead to a computational advantage

Solution Possibilities

1. Use sparse matrix for **G**
   together with `mest=G'*((G*G')\d)`
   (maybe damp a little, too)

2. Use analytic version of $\mathbf{GG}^T$
   together with `mest=G'*(GGT\d)`
   (maybe damp a little, too)

3. Use sparse matrix for **G**
   together with `bicg()` to solve $\mathbf{GG}^T\boldsymbol{\lambda}=\mathbf{d}$
   (maybe with a little damping, too)
   and then use $\mathbf{m}^{est}=\mathbf{G}^T\boldsymbol{\lambda}$

Solution Possibilities

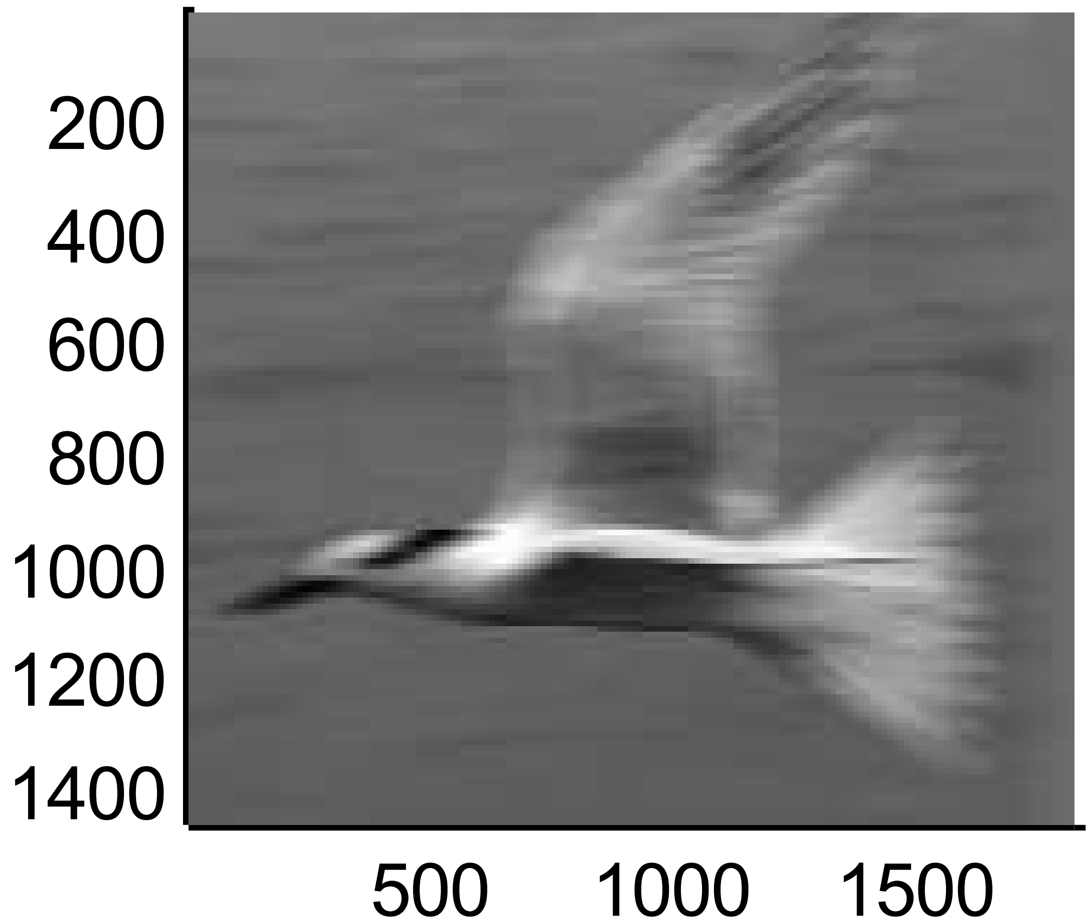
1. Use sparse matrix for **G**
   together with `mest=G'*((G*G')\d)`
   (maybe damp a little, too)

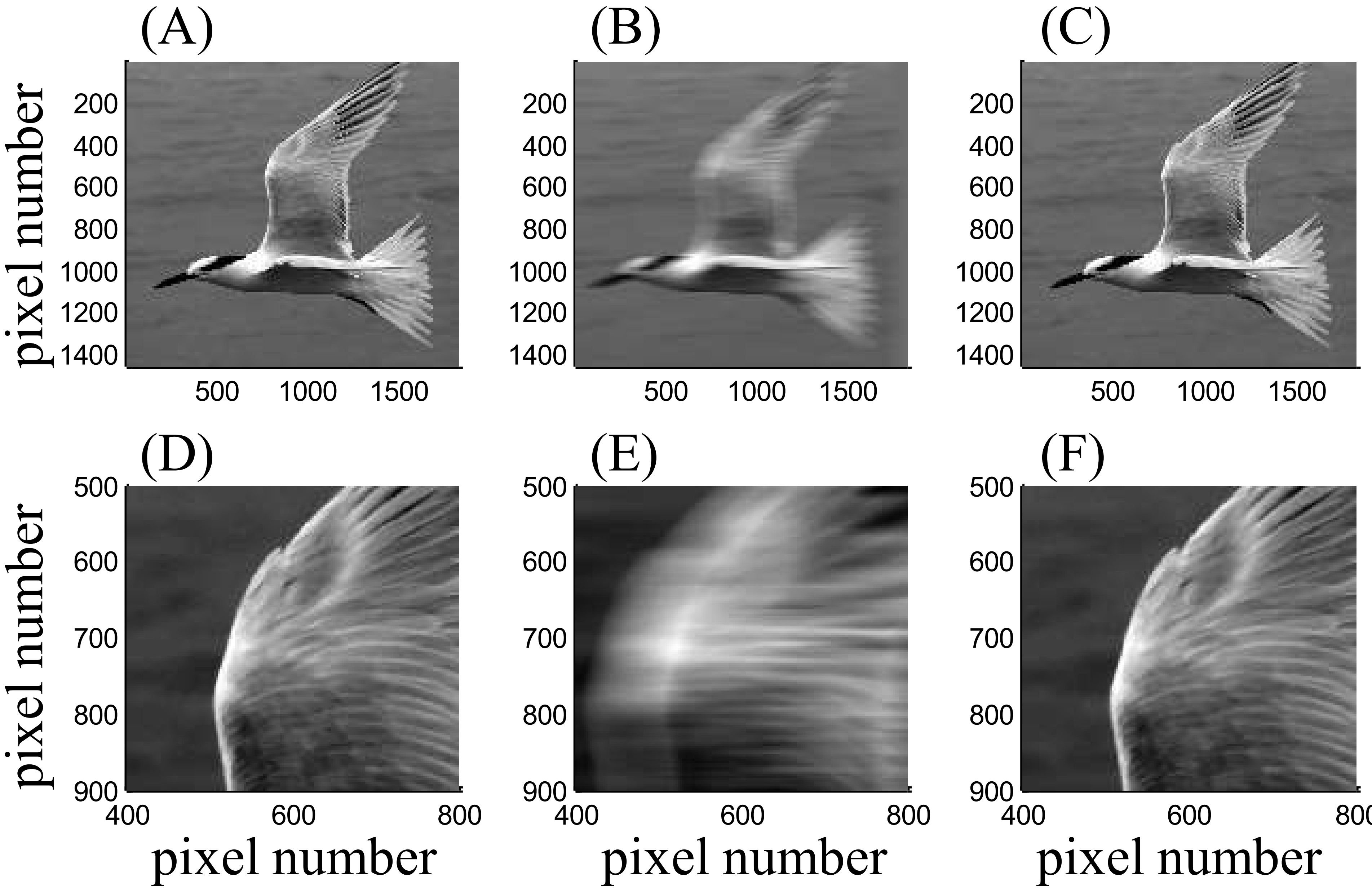
we used the simplest, which worked fine

2. Use analytic version of $\mathbf{GG}^T$
   together with `mest=G'*(GGT\d)`
   (maybe damp a little, too)

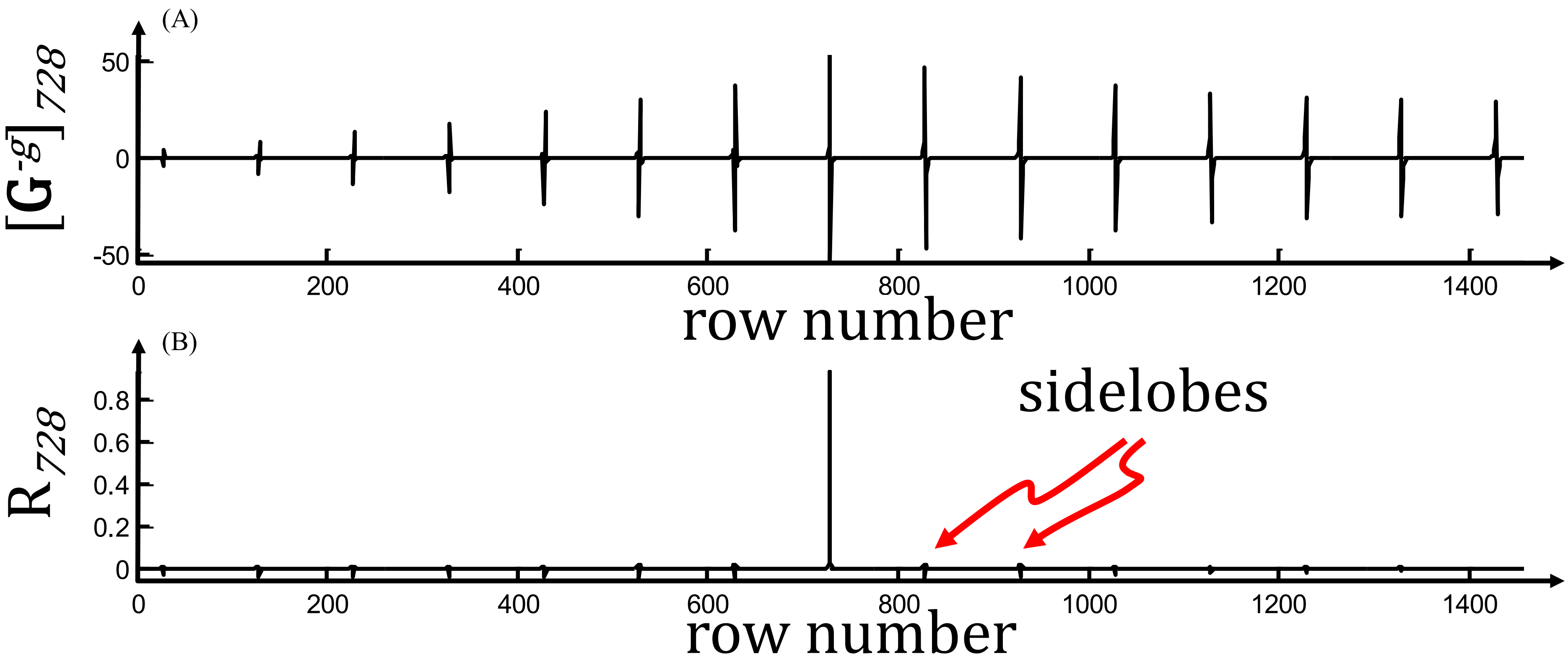
3. Use sparse matrix for **G**
   together with `bicg()` to solve $\mathbf{GG}^T\boldsymbol{\lambda}=\mathbf{d}$
   (maybe with a little damping, too)
   and then use $\mathbf{m}^{est}=\mathbf{G}^T\boldsymbol{\lambda}$

image blurred due to camera motion (100 point blur)





(A)
(B)
(C)
(D)
(E)
(F)
pixel number
pixel number
200
400
600
800
1000
1200
1400
500
1000
1500
500
600
700
800
900
400
600
800
pixel number
pixel number
pixel number


(A)
$[\mathbf{G}^{-g}]_{728}$
50
0
-50
0
200
400
600
800
1000
1200
1400
row number
(B)
R_{728}
0.8
0.6
0.4
0.2
0
sidelobes
0
200
400
600
800
1000
1200
1400
row number

Part 2

deconvolution filter

Convolution

general relationship for
linear systems
with translational invariance

Convolution

general relationship for
linear systems
with translational invariance

model $m(t)$
and
data $d(t)$
related by linear operator

Convolution

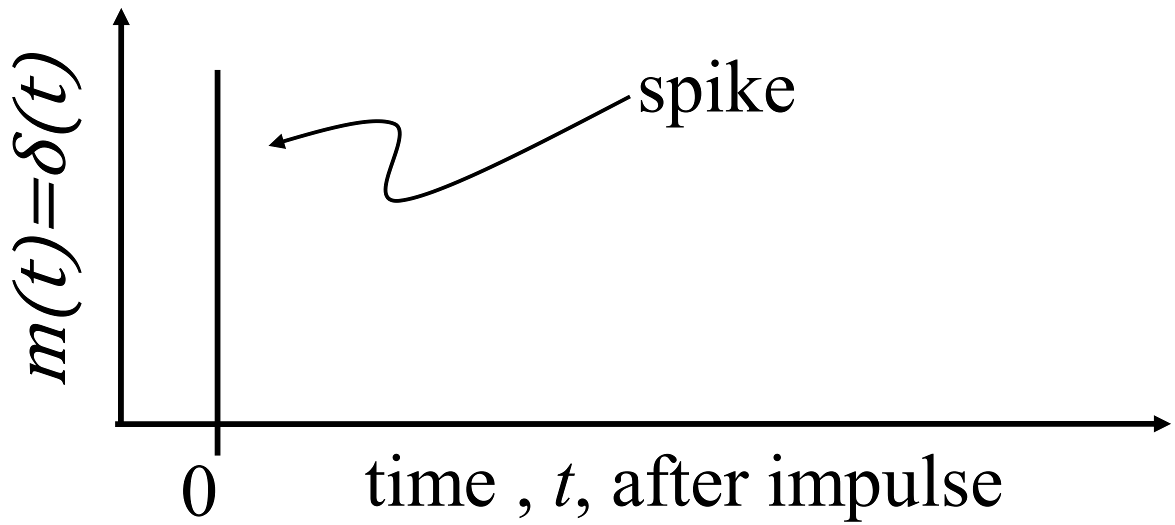
general relationship for
linear systems
with translational invariance

only relative time
matters

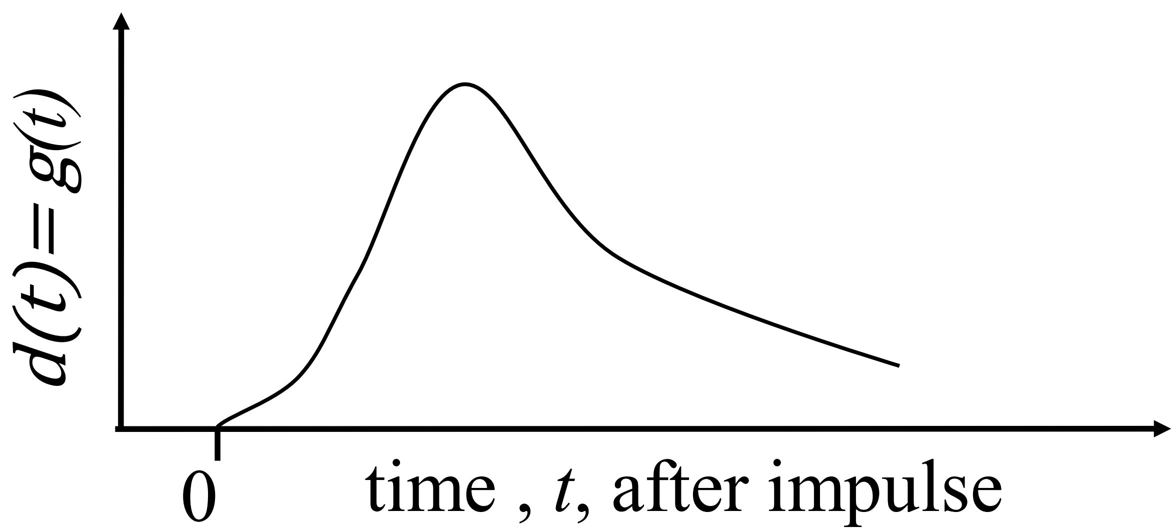
underlying principle

linear superposition

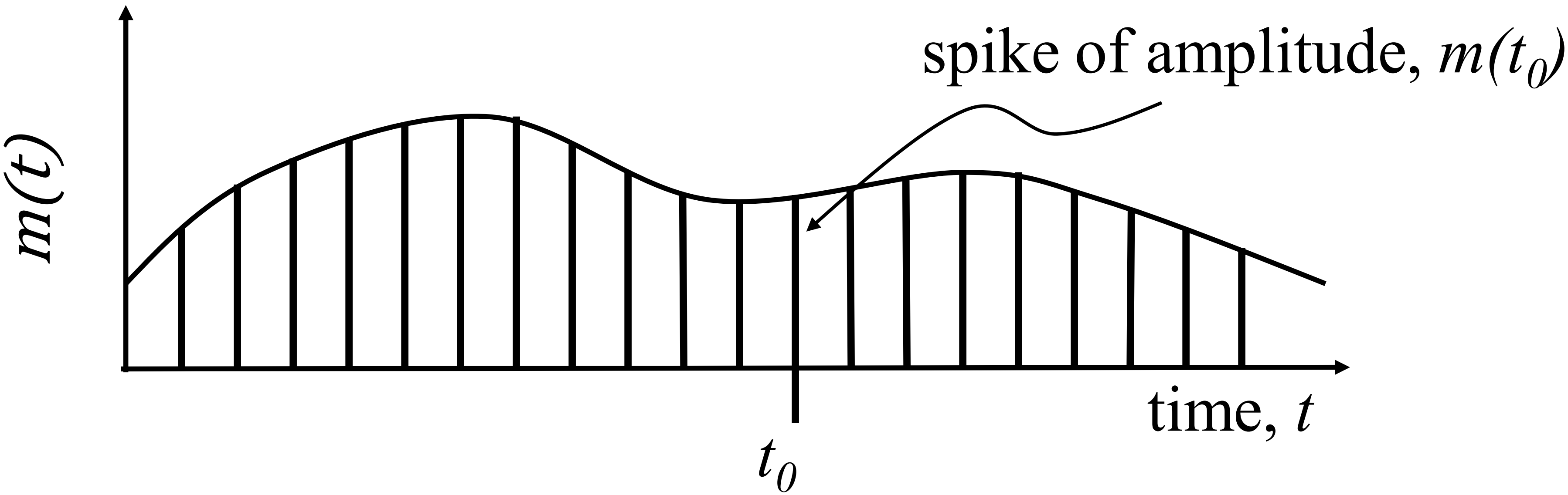
If the input of a spike $m(t)=\delta(t)$



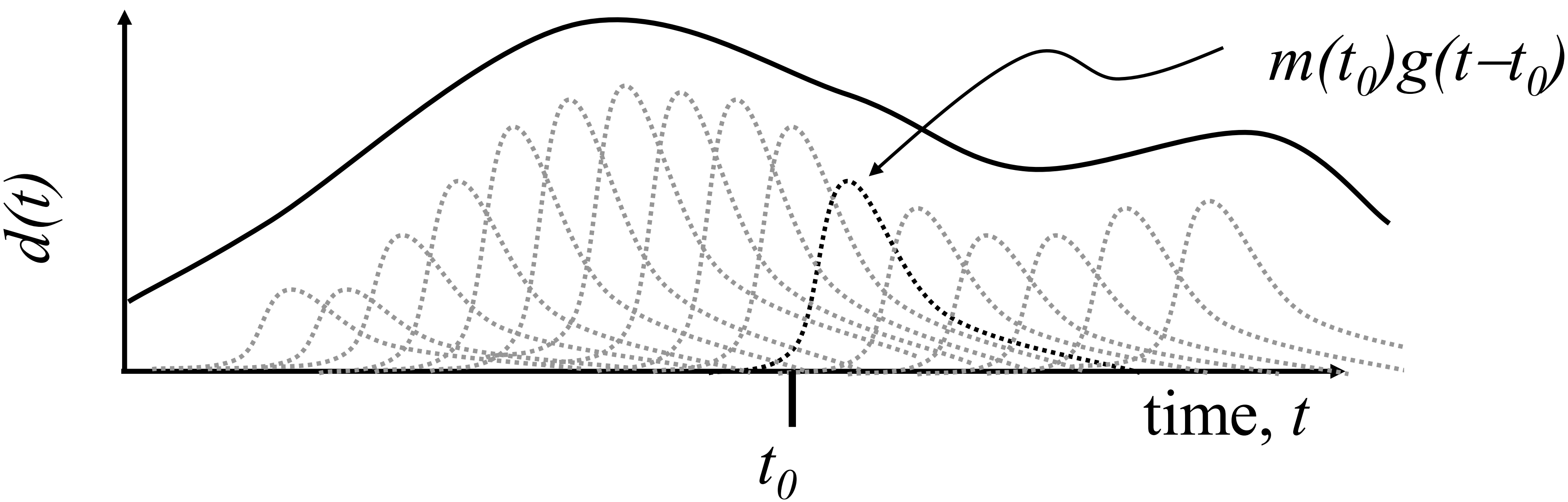
causes the output of $d(t)=g(t)$



Then the general input *m(t)*



causes the general output $d(t)=m(t)*g(t)$



convolution $d=m*g$

$$d(t) = m(t) * g(t) = \int g(t-\tau)\, m(\tau)\, \mathrm{d}\tau$$

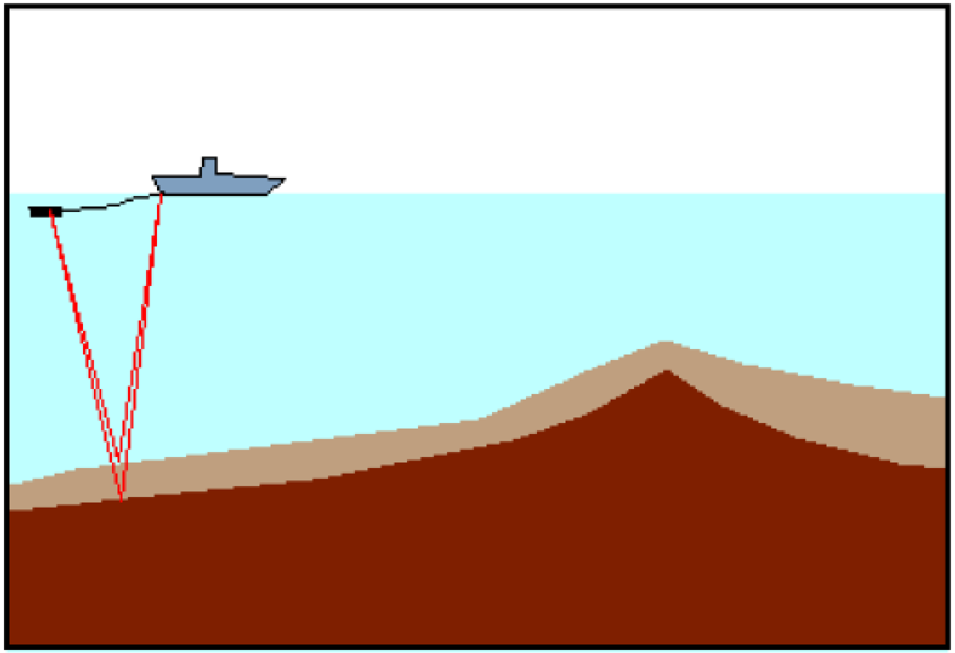
discrete convolution $d=m*g$

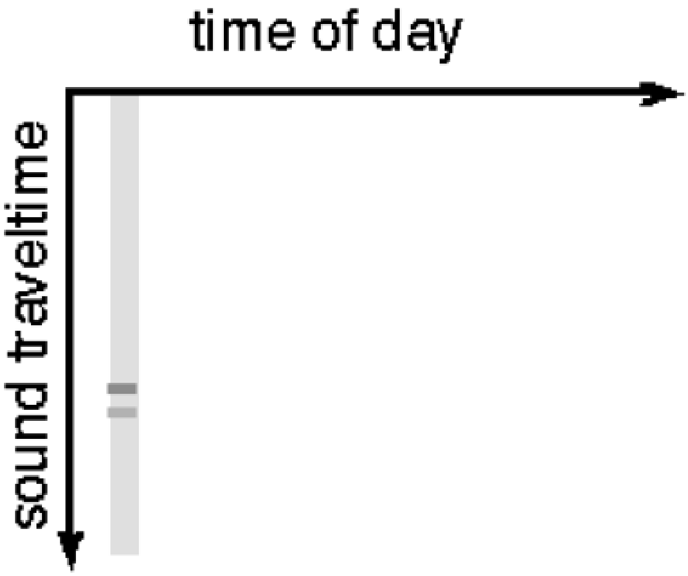
$$d_i = \Delta t \sum_{j=1}^{M} g_{i-j+1}\, m_j$$

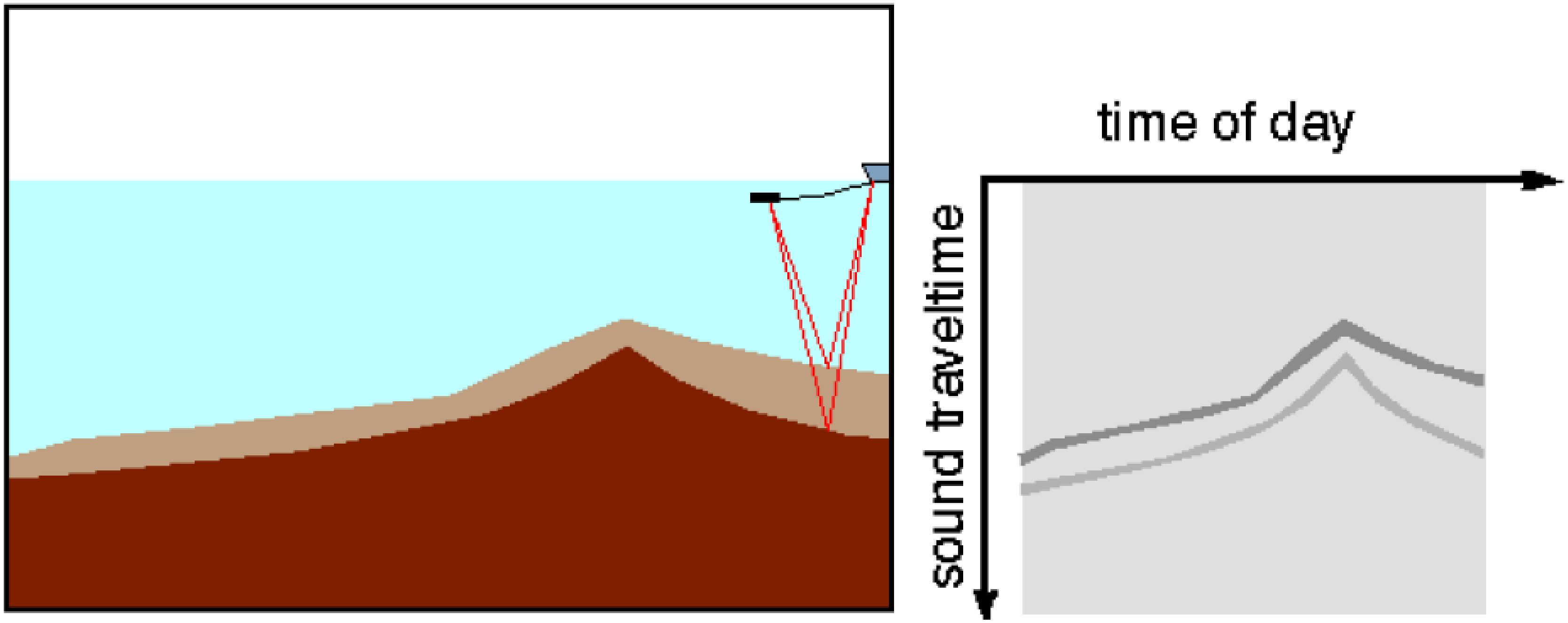
standard matrix from **d=Gm**

$$\mathbf{G} = \Delta t \begin{bmatrix} g_1 & 0 & 0 & \cdots & 0 \\ g_2 & g_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_N & g_{N-1} & g_{N-2} & \cdots & g_{N-M+1} \end{bmatrix}$$

seismic reflection sounding







time of day
sound traveltime

want airgun pulse to be as spiky as possible

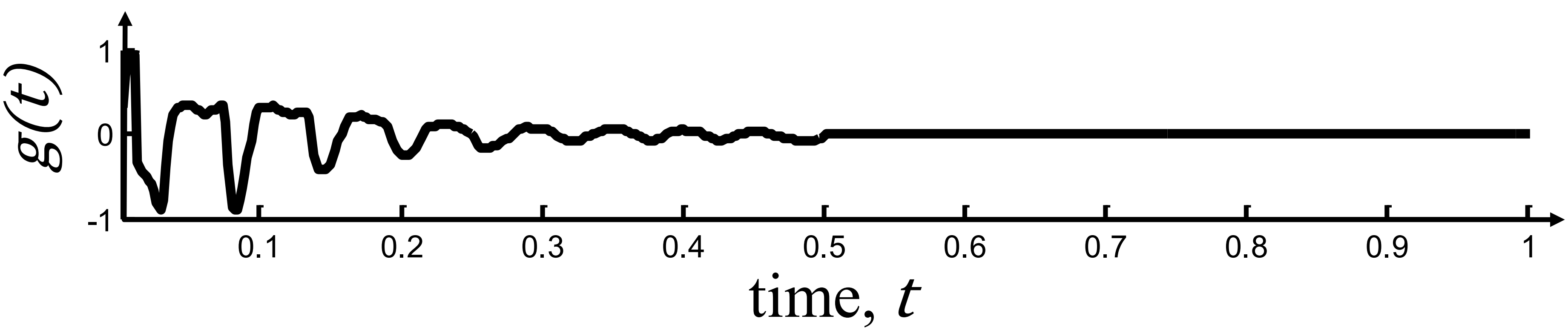
$$p(t) \quad = \quad g(t) \quad * \quad r(t)$$

pressure = airgun pulse * sea floor response

so as to be able to detect pulses in sea floor response

$$p(t) \approx r(t)$$

actual airgun pulse is ringy



so construct a *deconvolution filter* $m(t)$
so that

$$g(t) * m(t) = \delta(t)$$

and apply it to the data

$$p(t) = g(t) * r(t)$$

$$p(t) * m(t) = g(t) * m(t) * r(t) = r(t)$$

so construct a *deconvolution filter* $m(t)$ so that

$$\boxed{g(t) * m(t) = \delta(t)}$$

this is the equation we need to solve

and apply it to the data

$$p(t) = g(t)\, r(t)$$

$$\Downarrow$$

$$p(t) * m(t) = g(t) * m(t) * r(t) = r(t)$$

use discrete approximation of convolution

$$g(t) * m(t) = \delta(t)$$

$$\mathbf{Gm} = \mathbf{d}$$

discrete
approximation of
delta function

$$\Delta t \begin{bmatrix} g_1 & 0 & 0 & \cdots & 0 \\ g_2 & g_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_N & g_{N-1} & g_{N-2} & \cdots & g_{N-M+1} \end{bmatrix} \mathbf{m} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

solve with damped least squares

$$\mathbf{m}^{est} = [\mathbf{G}^T\mathbf{G} + \varepsilon^2\mathbf{I}]^{-1}\,\mathbf{G}^T\mathbf{d}$$

with $\mathbf{d} = [1, 0, 0, ..., 0]^T$
(or something similar)

matrices $\mathbf{G}^T\mathbf{G}$ and $\mathbf{G}^T\mathbf{d}$ can
be calculated analytically

$$\mathbf{G}^{\mathrm{T}}\mathbf{G} = (\Delta t)^2 \begin{bmatrix} \sum_{i=1}^{N} g_i^2 & \sum_{i=2}^{N-1} g_i g_{i-1} & \cdots \\ \sum_{i=2}^{N-1} g_i g_{i-1} & \sum_{i=1}^{N-1} g_i^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{G} = (\Delta t)^2 \begin{bmatrix} \sum_{i=1}^{N} g_i^2 & \sum_{i=2}^{N-1} g_i g_{i-1} & \cdots \\ \sum_{i=2}^{N-1} g_i g_{i-1} & \sum_{i=1}^{N-1} g_i^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

approximately Toeplitz with elements

$$[\mathbf{G}^{\mathrm{T}}\mathbf{G}]_{ij} = (\Delta t)^2 [\mathbf{g} \star \mathbf{g}]_{|i-j|+1}$$

$$\text{where} \quad [\mathbf{a} \star \mathbf{b}]_i = \sum_{j=1}^{N} a_j b_{i+j-1}$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{G} = (\Delta t)^2 \begin{bmatrix} \sum_{i=1}^{N} g_i^2 & \sum_{i=2}^{N-1} g_i g_{i-1} & \cdots \\ \sum_{i=2}^{N-1} g_i g_{i-1} & \sum_{i=1}^{N-1} g_i^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

approximately Toeplitz with elements

$$[\mathbf{G}^{\mathrm{T}}\mathbf{G}]_{ij} = (\Delta t)^2 [\mathbf{g} \star \mathbf{g}]_{|i-j|+1}$$

autocorrelation of **g**

where $[\mathbf{a} \star \mathbf{b}]_i = \sum_{j=1}^{N} a_j b_{i+j-1}$

$$\mathbf{G}^{\mathrm{T}}\mathbf{d} = \Delta t \begin{bmatrix} \sum_{i=1}^{N} d_i g_i \\ \sum_{i=2}^{N-1} d_i g_{i-1} \\ \vdots \end{bmatrix}$$

$$[\mathbf{G}^{\mathrm{T}}\mathbf{d}]_i = \Delta t \, [\mathbf{g} \star \mathbf{d}]_i$$

where $[\mathbf{a} \star \mathbf{b}]_i = \sum_{j=1}^{N} a_j b_{i+j-1}$

$$\mathbf{G}^{\mathrm{T}}\mathbf{d} = \Delta t \begin{bmatrix} \sum_{i=1}^{N} d_i g_i \\ \sum_{i=2}^{N-1} d_i g_{i-1} \\ \vdots \end{bmatrix}$$

$$[\mathbf{G}^{\mathrm{T}}\mathbf{d}]_i = \Delta t \, [\mathbf{g} \star \mathbf{d}]_i$$

cross-correlation of **g** and **d**

where $[\mathbf{a} \star \mathbf{b}]_i = \sum_{j=1}^{N} a_j b_{i+j-1}$

Solution Possibilities

1. Use sparse matrix for $\mathbf{G}$
   together with `mest=(G'*G)\(G'*d)`
   (maybe damping a little, too)

2. Use analytic versions of $\mathbf{G}^T\mathbf{G}$ and $\mathbf{G}^T\mathbf{d}$
   together with `mest=GTG\GTd`
   (maybe damp a little, too)

3. Never form $\mathbf{G}$, just work with its columns, $\mathbf{g}$
   use `bicg()` to solve $\mathbf{G}^T\mathbf{G}\,\mathbf{m} = \mathbf{G}^T\mathbf{d}$
   but use `conv()` to compute $\mathbf{G}^T(\mathbf{G}\mathbf{v})$

4. Same as 3 but add a priori information of
   smoothness

Solution Possibilities

1. Use sparse matrix for **G**
   together with `mest=(G'*G)\(G'*d)`
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   use `bicg()` to solve $\mathbf{G}^T\mathbf{G}\,\mathbf{m} = \mathbf{G}^T\mathbf{d}$
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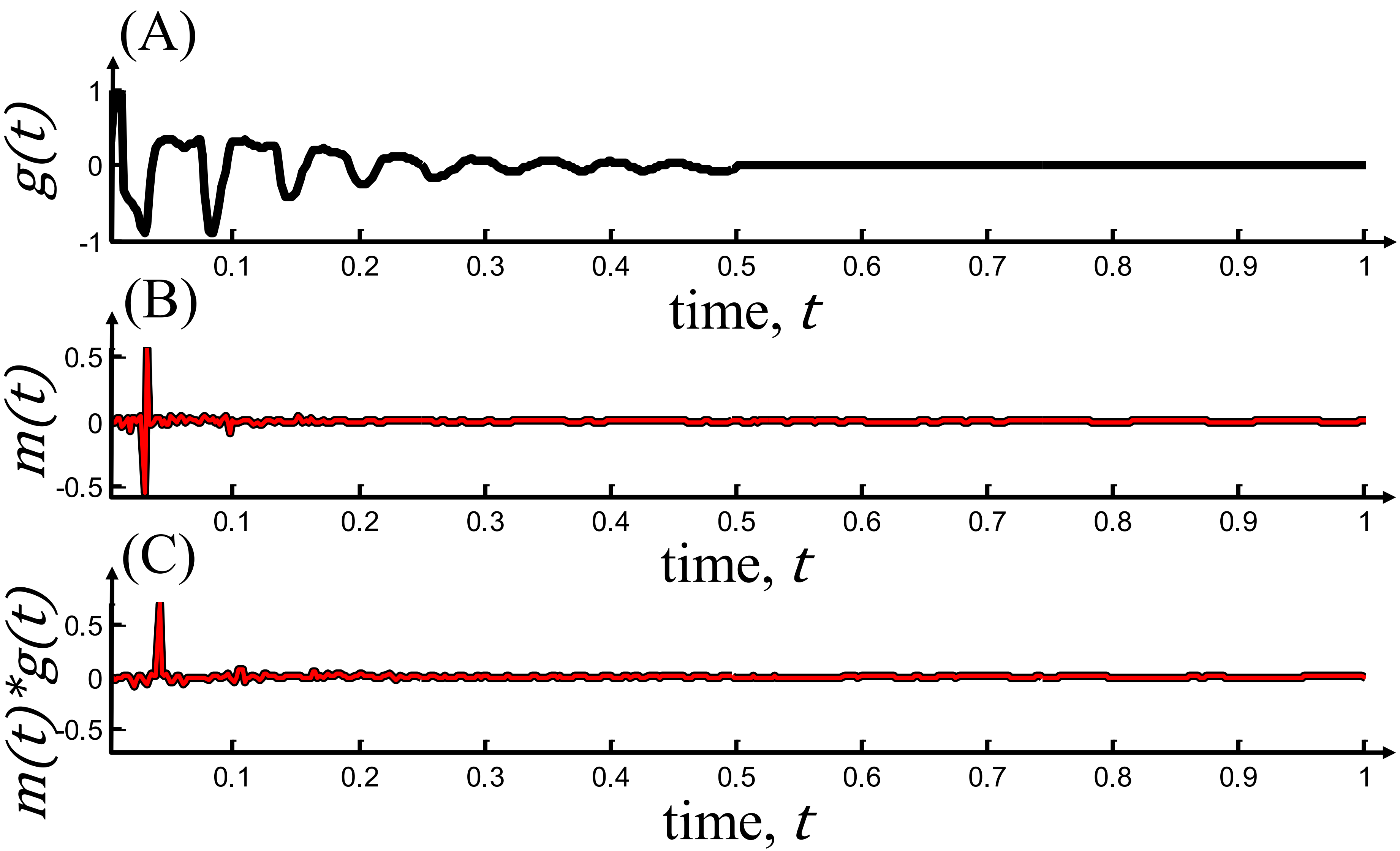
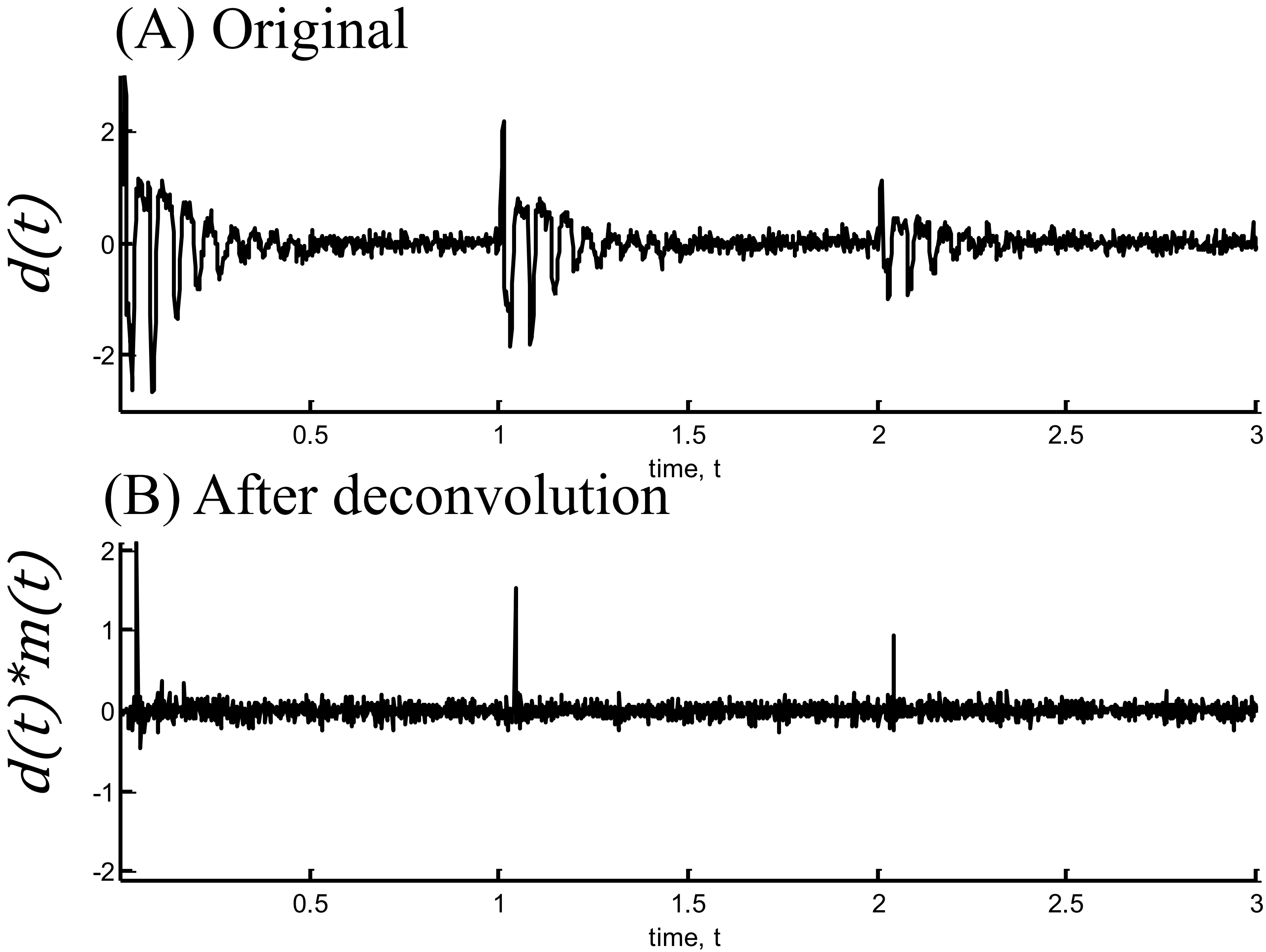
4. Same as 3 but add a priori information of smoothness

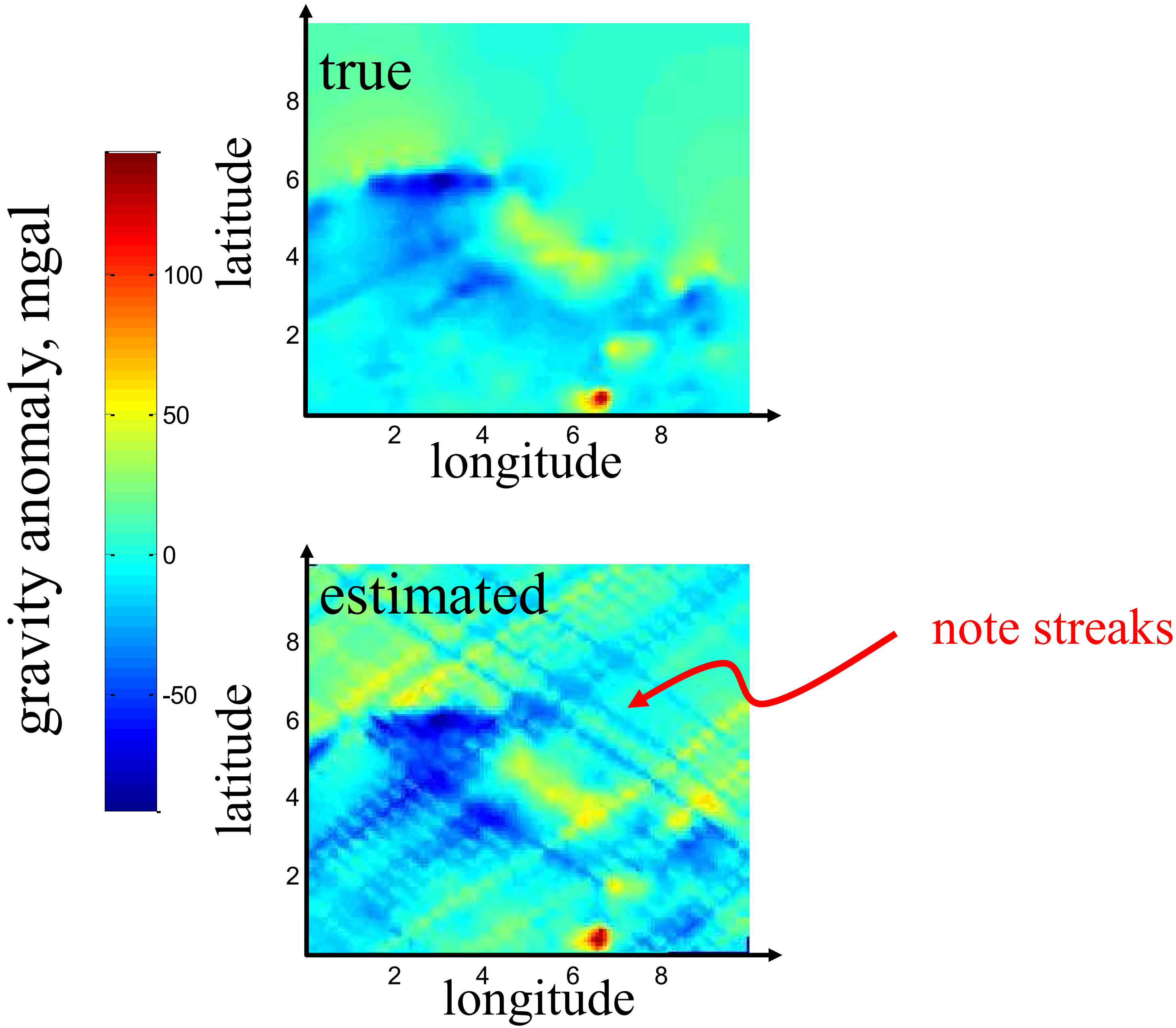
we used this complicated but very fast method


(A)
g(t)
1
0
-1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time, t
(B)
m(t)
0.5
0
-0.5
time, t
(C)
m(t)*g(t)
0.5
0
-0.5
time, t


(A) Original
d(t)
2
0
-2
0.5
1
1.5
2
2.5
3
time, t
(B) After deconvolution
d(t)*m(t)
2
1
0
-1
-2
0.5
1
1.5
2
2.5
3
time, t

Part 3

minimization of cross-over errors


gravity anomaly, mgal
100
50
0
-50
true
latitude
8
6
4
2
longitude
2
4
6
8
estimated
latitude
8
6
4
2
longitude
2
4
6
8
note streaks

general idea

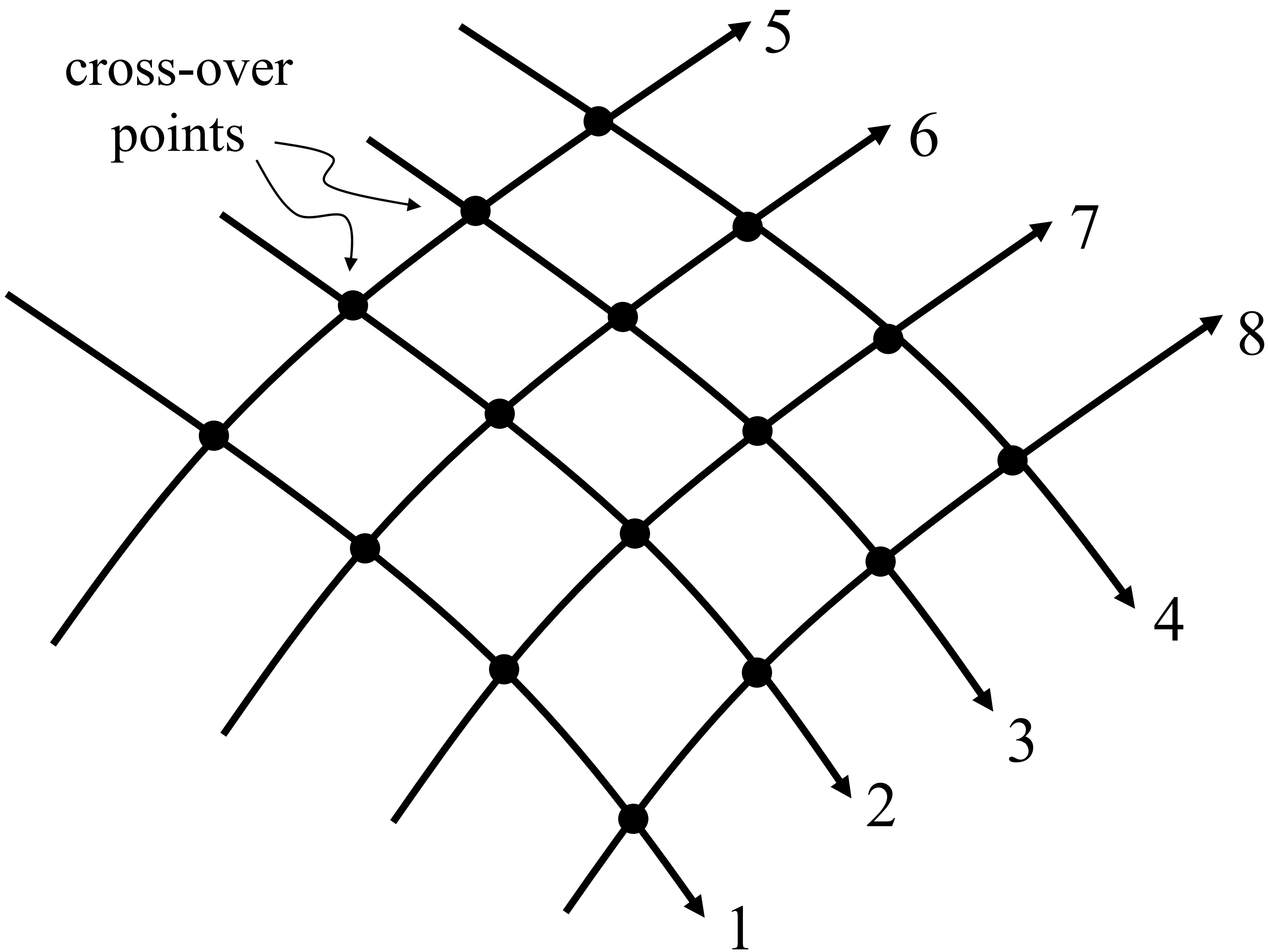
data ***s*** is measured along tracks

data along each track is off by an additive constant

theory

$$s_j^{\text{obs (track i)}} = s_j^{\text{true (track i)}} + m^{\text{(track i)}}$$

goal is to estimate the constants by minimizing the error at track intersections



cross-over
points
5
6
7
8
4
3
2
1

ith intersection has
ascending track A_i and descending track D_i

$$s_{Ai}^{obs} = s_{Ai}^{true} + m_{Ai}$$

$$s_{Di}^{obs} = s_{Di}^{true} + m_{Di}$$

subtract

$$s_{Ai}^{obs} - s_{Di}^{obs} = m_{Ai} - m_{Di}$$

has form

$$\mathbf{d} = \mathbf{G}\mathbf{m}$$

the matrix **G** is very sparse

$$G_{ij} = \delta_{jA_i} - \delta_{jD_i}$$

every row is all zeros, except for a single +1 and a single -1 note that this problem has an inherent non-uniqueness

m is determined only to an overall additive constant

one possibility is to use damped least squares, to choose the smallest **m**

(you can always add a constant later)

the matrices $\mathbf{G}^{\mathrm{T}}\mathbf{G}$ and $\mathbf{G}^{\mathrm{T}}\mathbf{d}$ can be calculated semi-analytically

$$[\mathbf{G}^{\mathrm{T}}\mathbf{G}]_{rs} = \sum_{i=1}^{N} G_{ir} G_{is} = \sum_{i=1}^{N} \left(\delta_{rA_i} - \delta_{rD_i}\right)\left(\delta_{sA_i} - \delta_{sD_i}\right)$$

$$= \sum_{i=1}^{N} \left(\delta_{rA_i}\delta_{sA_i} - \delta_{rA_i}\delta_{sD_i} - \delta_{rD_i}\delta_{sA_i} + \delta_{rD_i}\delta_{sD_i}\right)$$

$$[\mathbf{G}^{\mathrm{T}}\mathbf{G}]_{rr} = \sum_{i=1}^{N} \left(\delta_{rA_i}\delta_{rA_i} - 2\delta_{rA_i}\delta_{rD_i} + \delta_{rD_i}\delta_{rD_i}\right)$$

$$[\mathbf{G}^{\mathrm{T}}\mathbf{d}]_r = \sum_{i=1}^{N} G_{ir} d_i = \sum_{i=1}^{N} \left(\delta_{rA_i} - \delta_{rD_i}\right) d_i$$

recipe starting with zeroed $\mathbf{G}^T\mathbf{G}$ and $\mathbf{G}^T\mathbf{d}$

(1) Add *1* to the $r = A_i$, $s = A_i$ element of $[\mathbf{G}^T\mathbf{G}]_{rs}$.

(2) Add *1* to the $r = D_i$, $s = D_i$ element of $[\mathbf{G}^T\mathbf{G}]_{rs}$.

(3) Subtract *1* from the $r = A_i$, $s = D_i$ element of $[\mathbf{G}^T\mathbf{G}]_{rs}$.

(4) Subtract *1* from the $r = D_i$, $s = A_i$ element of $[\mathbf{G}^T\mathbf{G}]_{rs}$.

(5) Add d_i to the $r = A_i$ element of $[\mathbf{G}^T\mathbf{d}]_r$.

(6) Subtract d_i from the $r = D_i$ element of $[\mathbf{G}^T\mathbf{d}]_r$.

Solution Possibilities

1. Use sparse matrix for **G**
   together with damped least squares
   `mest=(G'*G+e2*speye(M,M))\(G'*d)`

2. Use analytic versions of $\mathbf{G}^T\mathbf{G}$ and $\mathbf{G}^T\mathbf{d}$
   add damping directly to the diagonal of $\mathbf{G}^T\mathbf{G}$
   then use `mest=GTGpe2I\GTd`

3. Use sparse matrix for **G**
   together with **bicg()** version of damped least squares

4. Methods 1 or 2, but use hard constraint instead of damping to implement $\Sigma_i\ m_i = 0$

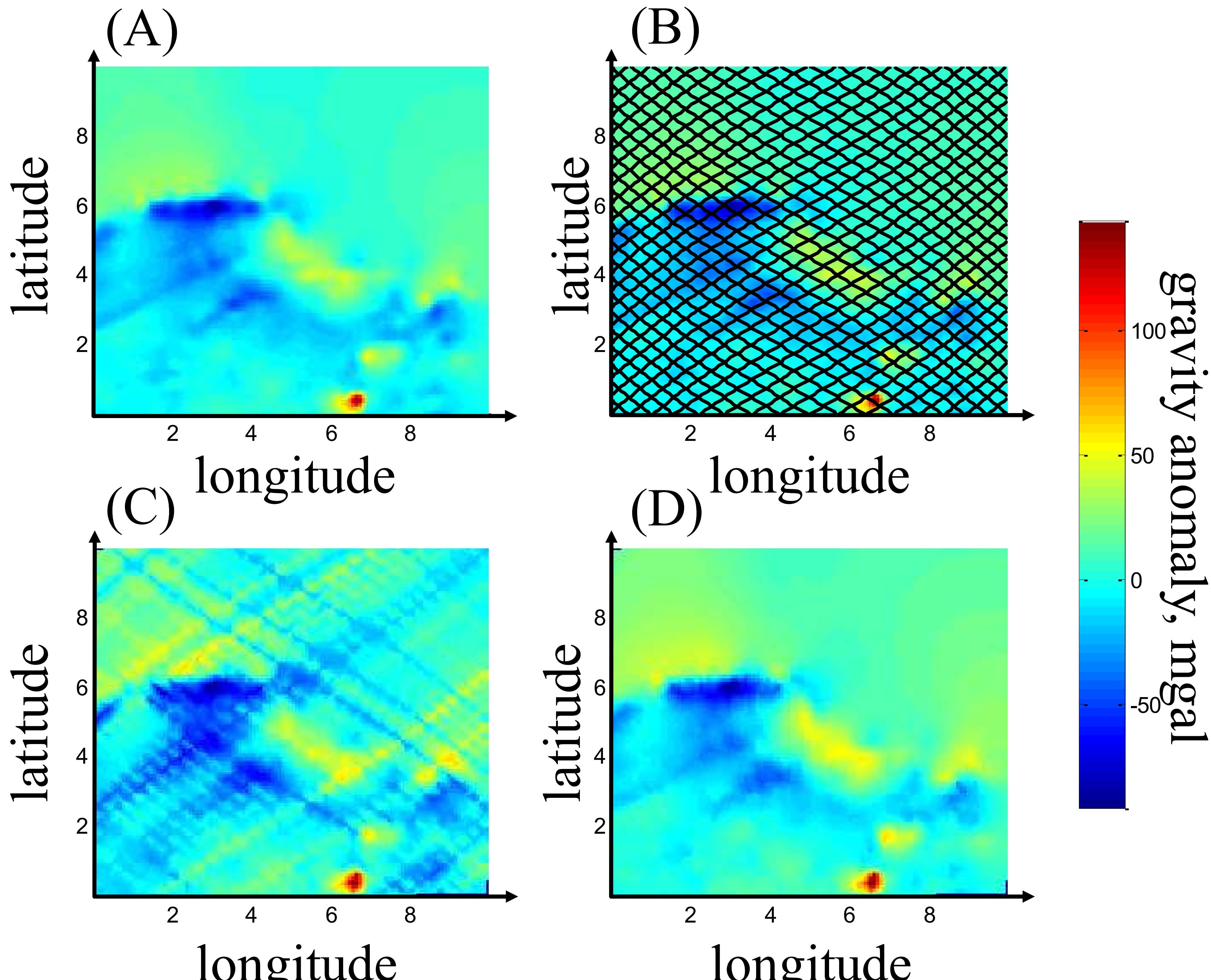
Solution Possibilities

1. Use sparse matrix for **G**
   together with damped least squares
   `mest=(G′*G*e2*speye(M,M))\(G′*d)`

2. Use analytic versions of $\mathbf{G}^{T}\mathbf{G}$ and $\mathbf{G}^{T}\mathbf{d}$
   add damping directly to the diagonal of $\mathbf{G}^{T}\mathbf{G}$
   then use `mest=GTG\GTd`

3. Use sparse matrix for **G** our choice
   together with `bicg()` version of damped least squares

4. Methods 1 or 2, but use hard constraint instead of damping


(A)
(B)
(C)
(D)
latitude
longitude
2
4
6
8
gravity anomaly, mgal
100
50
0
-50