Lecture 23

Exemplary Inverse Problems including Earthquake Location

Syllabus

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Lecture 03	Probability and Measurement Error, Part 2
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Lecture 06	Resolution and Generalized Inverses
Lecture 07	Backus-Gilbert Inverse and the Trade Off of Resolution and Variance
Lecture 08	The Principle of Maximum Likelihood
Lecture 09	Inexact Theories
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Lecture 14	Nonlinear Problems: Grid and Monte Carlo Searches
Lecture 15	Nonlinear Problems: Newton's Method
Lecture 16	Nonlinear Problems: Simulated Annealing and Bootstrap Confidence Intervals
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Lecture 20	Linear Operators and Their Adjoints
Lecture 21	Fréchet Derivatives
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Lecture 23	**Exemplary Inverse Problems, incl. Earthquake Location**
Lecture 24	Exemplary Inverse Problems, incl. Vibrational Problems

Purpose of the Lecture

solve a few exemplary inverse problems

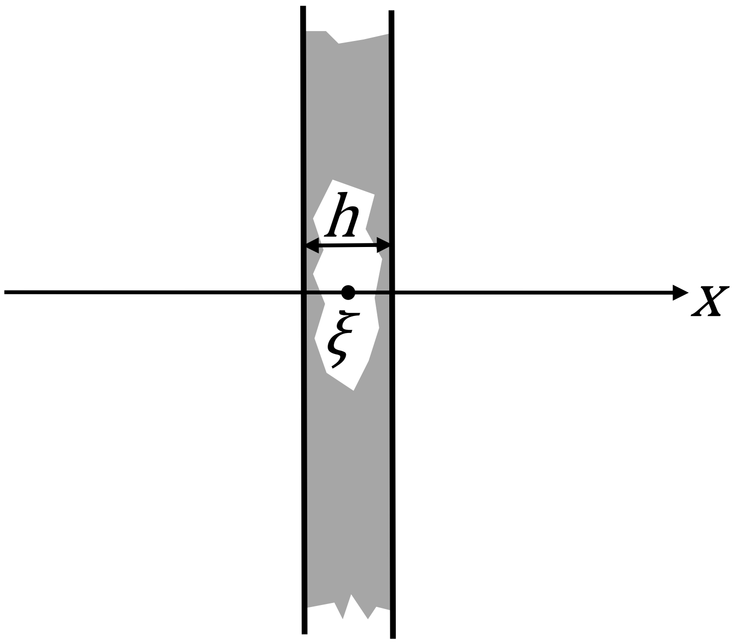
thermal diffusion
earthquake location
fitting of spectral peaks

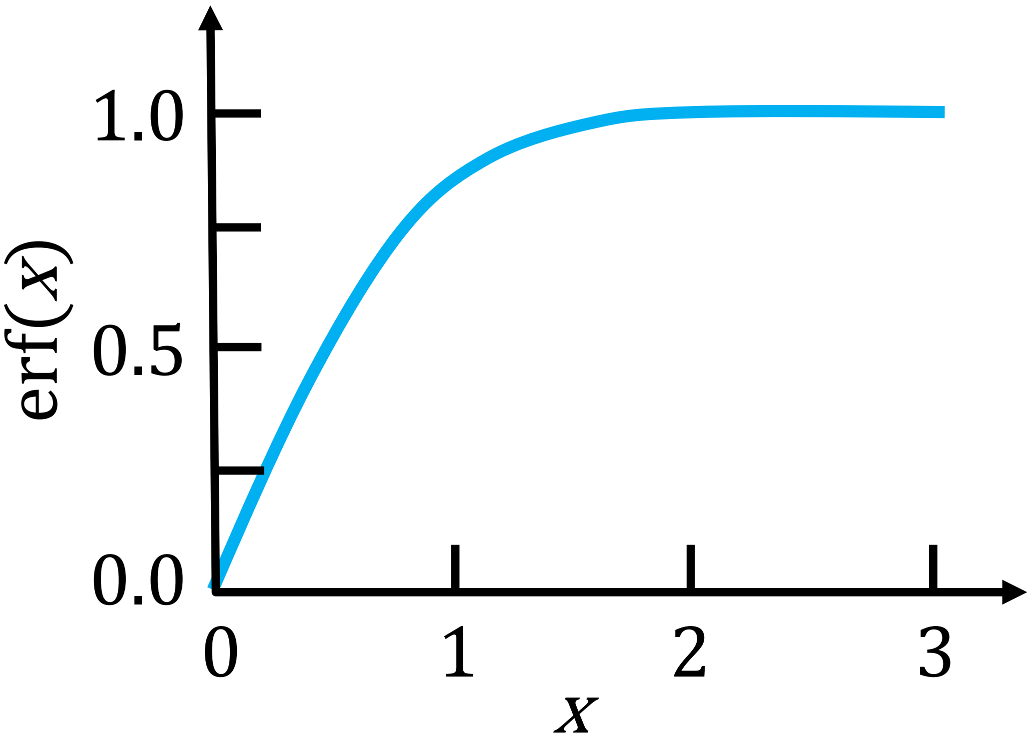
Part 1

thermal diffusion

temperature in a cooling slab

$$T(x,t) = \tfrac{1}{2}T_0\left\{\operatorname{erf}\left[\frac{x-(\xi-\tfrac{1}{2}h)}{\sqrt{t}}\right]-\operatorname{erf}\left[\frac{x-(\xi+\tfrac{1}{2}h)}{\sqrt{t}}\right]\right\} = T_0\, g(x,t,\xi)$$





temperature due to M cooling slabs (use linear superposition)

$$T(x_i, t) = \sum_{j=1}^{M} g(x_i, t, \xi_j)\ m_j$$

temperature due to M slabs each with initial temperature m_j

$$T(x_i, t) = \sum_{j=1}^{M} g(x_i, t, \xi_j)\ m_j$$

temperature measured at time $t>0$

initial temperature

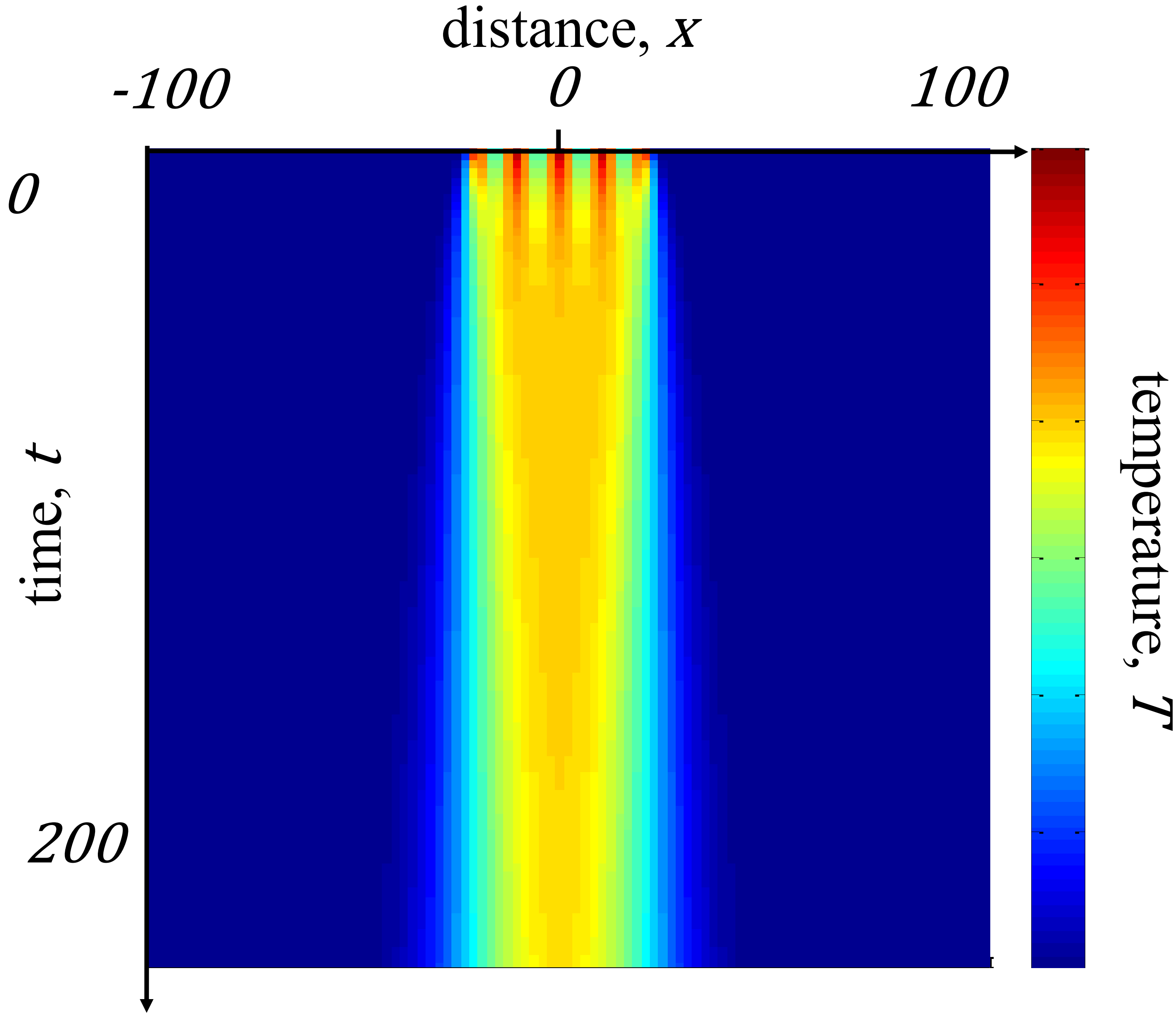
inverse problem
infer initial temperature **m**
using temperatures measures at a suite of *x*s
at some fixed later time t

$$T(x_i, t) = \sum_{j=1}^{M} g(x_i, t, \xi_j)\ m_j$$

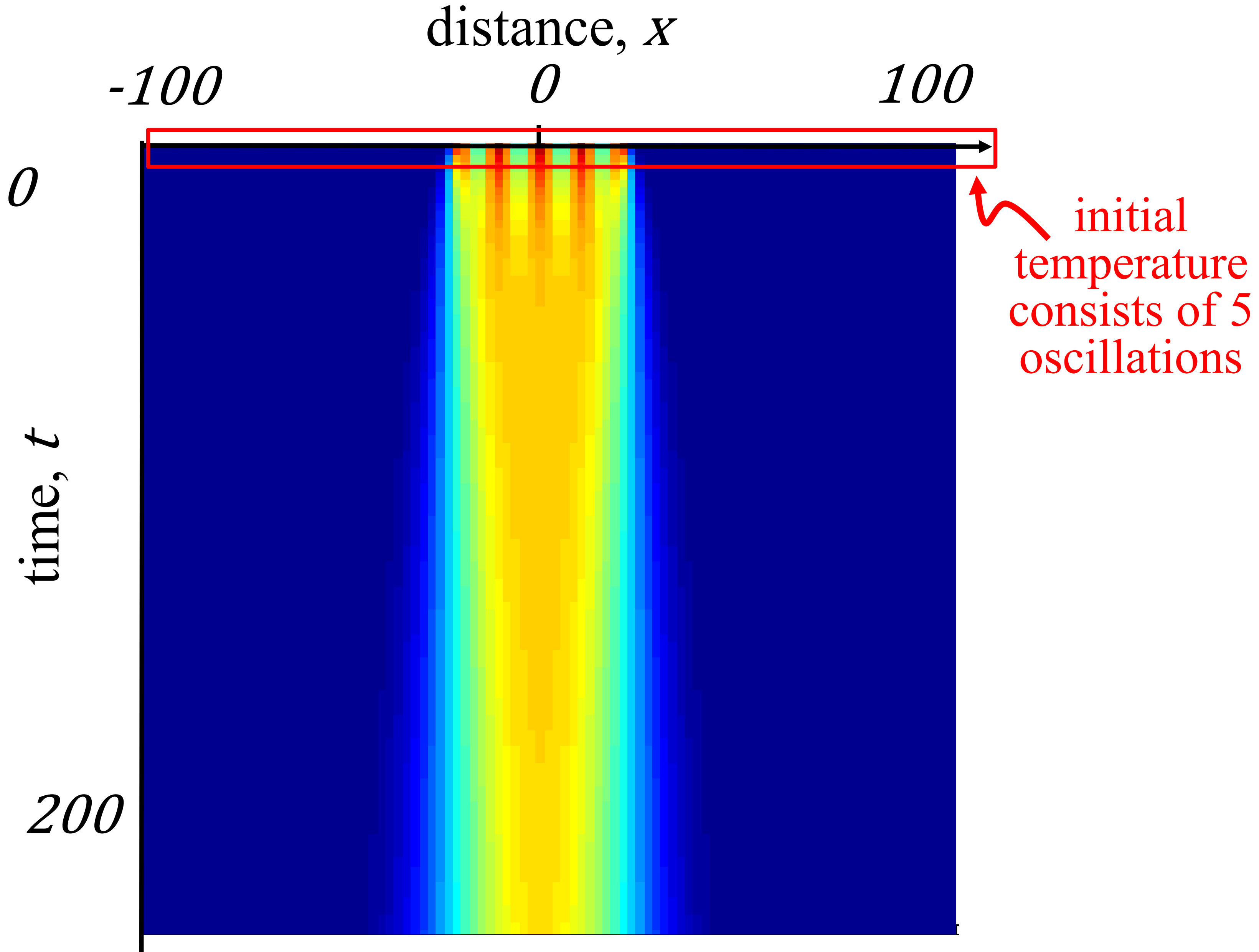
data

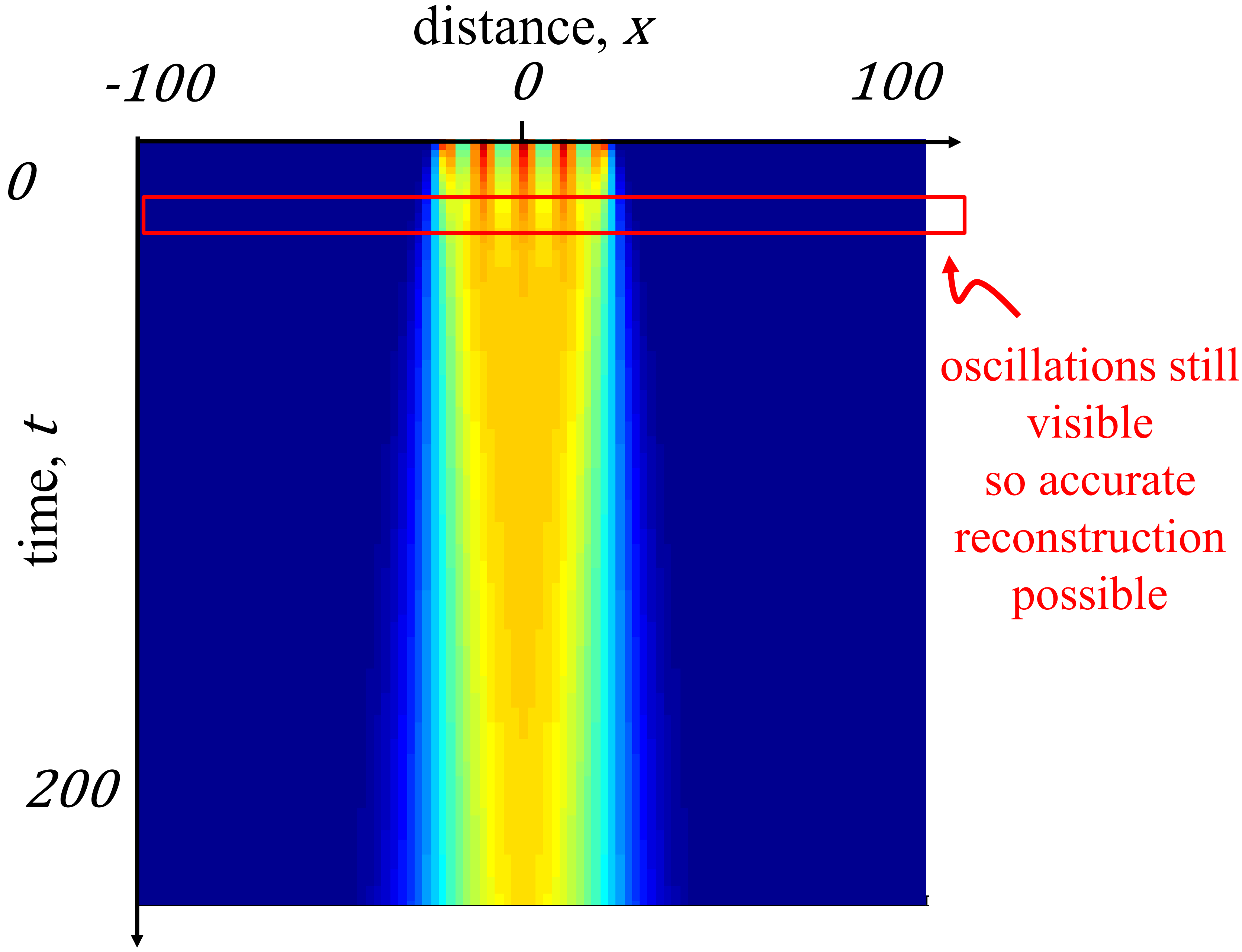
model parameters

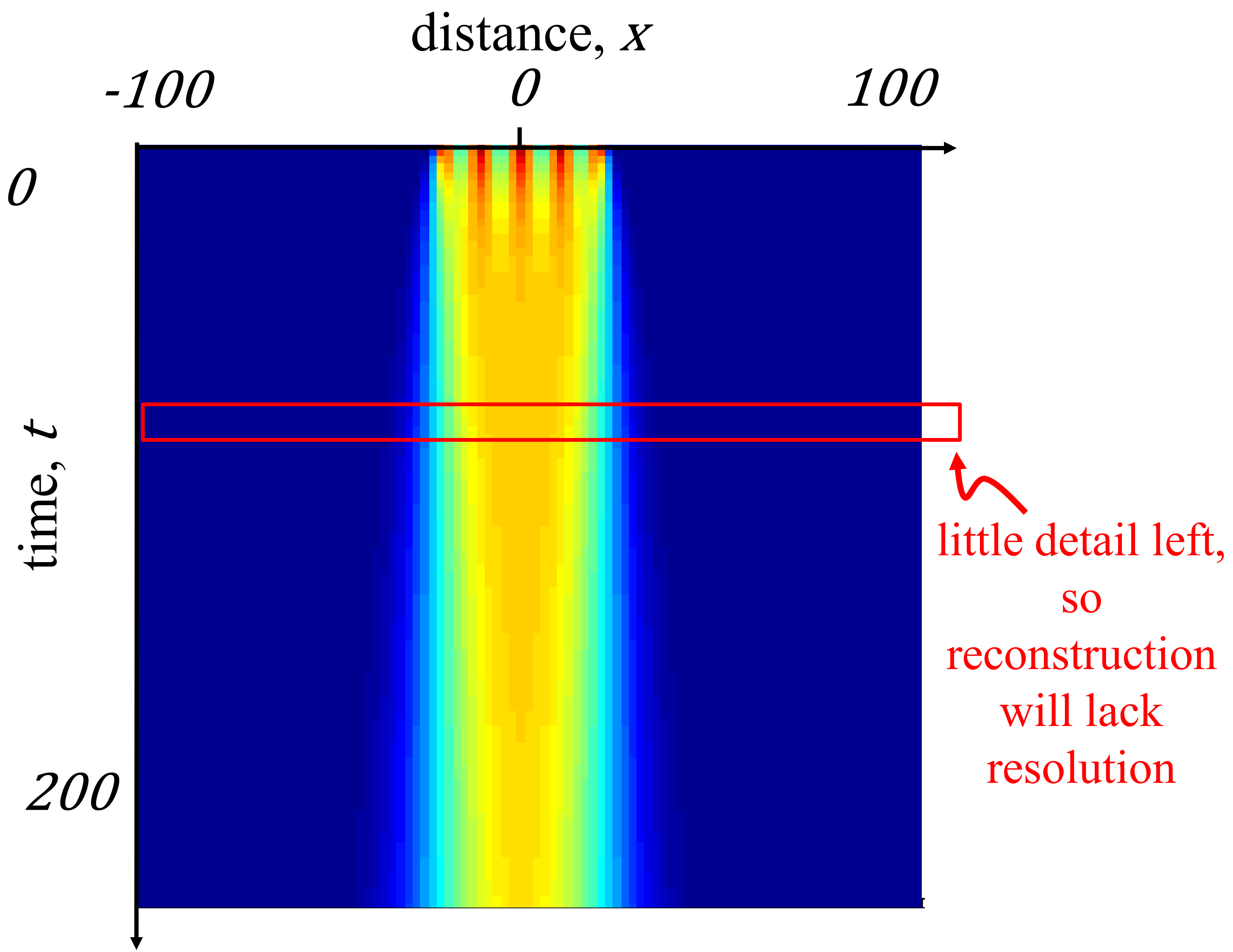
$$\mathbf{d} = \mathbf{G}\,\mathbf{m}$$



distance, x
-100
0
100
0
200
time, t
temperature, T


distance, x
-100
0
100
0
time, t
200
initial temperature consists of 5 oscillations


distance, x
-100
0
100
0
time, t
200
oscillations still visible so accurate reconstruction possible


distance, x
-100
0
100
0
time, t
200
little detail left,
so
reconstruction
will lack
resolution

What Method ?

The resolution is likely to be rather poor, especially when data are collected at later times

damped least squares

$$\mathbf{G}^{-g} = [\mathbf{G}^T\mathbf{G}+\varepsilon^2\mathbf{I}]^{-1}\mathbf{G}^T$$

damped minimum length

$$\mathbf{G}^{-g} = \mathbf{G}^T\,[\mathbf{G}\mathbf{G}^T+\varepsilon^2\mathbf{I}]^{-1}$$

Backus-Gilbert

What Method ?

The resolution is likely to be rather poor, especially when data are collected at later times

damped least squares

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damped minimum length

$$\mathbf{G}^{-g} = \mathbf{G}^T [\mathbf{G}\mathbf{G}^T+\varepsilon^2\mathbf{I}]^{-1}$$

actually, these generalized inverses are equal

Backus-Gilbert

What Method ?

The resolution is likely to be rather poor, especially when data are collected at later times

damped least squares

$$\mathbf{G}^{-g} = [\mathbf{G}^T\mathbf{G}+\varepsilon^2\mathbf{I}]^{-1}\mathbf{G}^T$$

damped minimum length

$$\mathbf{G}^{-g} = \mathbf{G}^T [\mathbf{G}\mathbf{G}^T+\varepsilon^2\mathbf{I}]^{-1}$$

Backus-Gilbert ← might produce solutions with fewer artifacts

Try both

damped least squares

Backus-Gilbert

Solution Possibilities

1. Damped Least Squares:
   Matrix **G** is not sparse
   no analytic version of $\mathbf{G}^T\mathbf{G}$ is available
   M=100 is rather small
   experiment with values of ε^2
   ```
   mest=(G'*G+e2*eye(M,M))\(G'*d)
   ```

2. Backus-Gilbert
   use standard formulation, with damping α
   experiment with values of α
   ```
   GMG = zeros(M,N);
   u = G*ones(M,1);
   for k = [1:M]
       S = G * diag(([1:M]-k).^2) * G';
       Sp = alpha*S + (1-alpha)*eye(N,N);
       uSpinv = u'/Sp;
       GMG(k,:) = uSpinv / (uSpinv*u);
   end
   ```

Solution Possibilities

1. Damped Least Squares:
   Matrix **G** is not sparse
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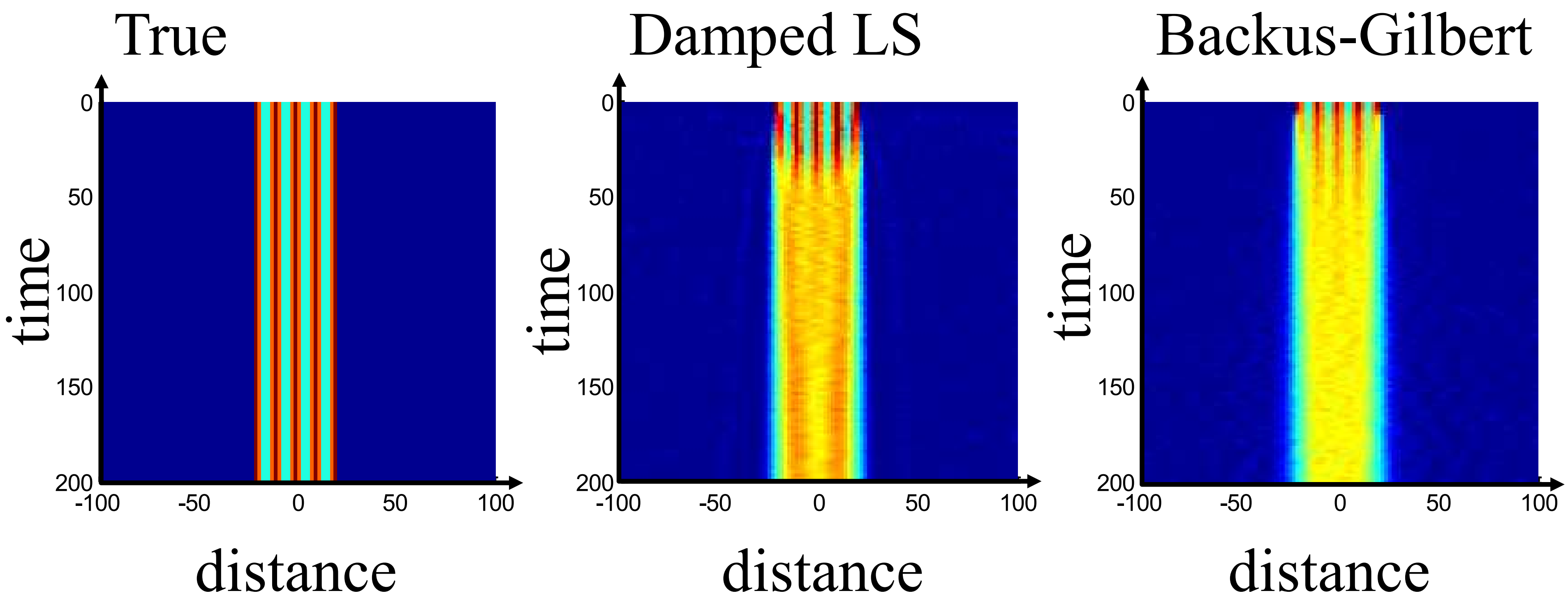
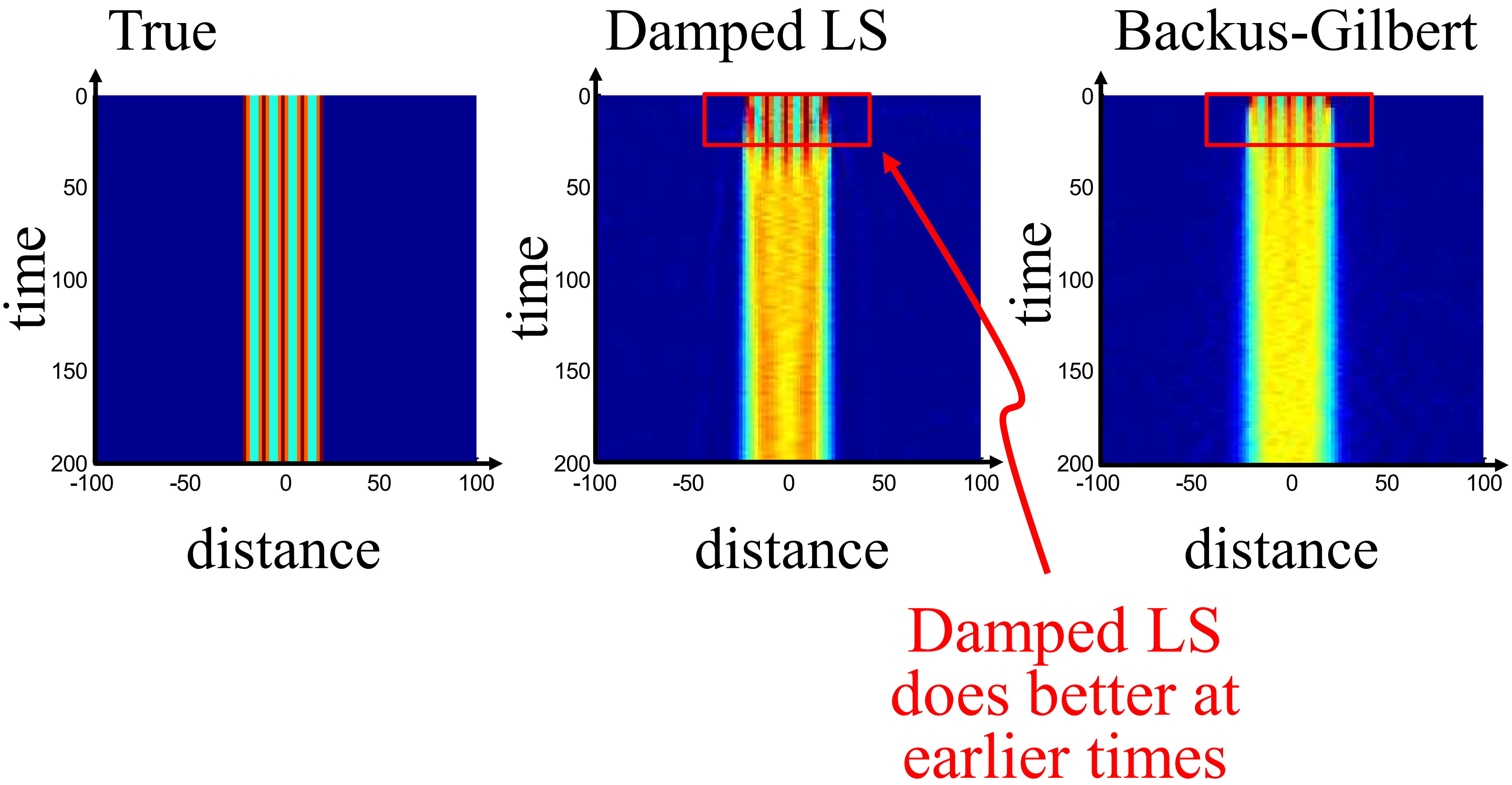
    uSpinv = u'/Sp;
    GMG(k,:) = uSpinv / (uSpinv*u);
end
```

try both

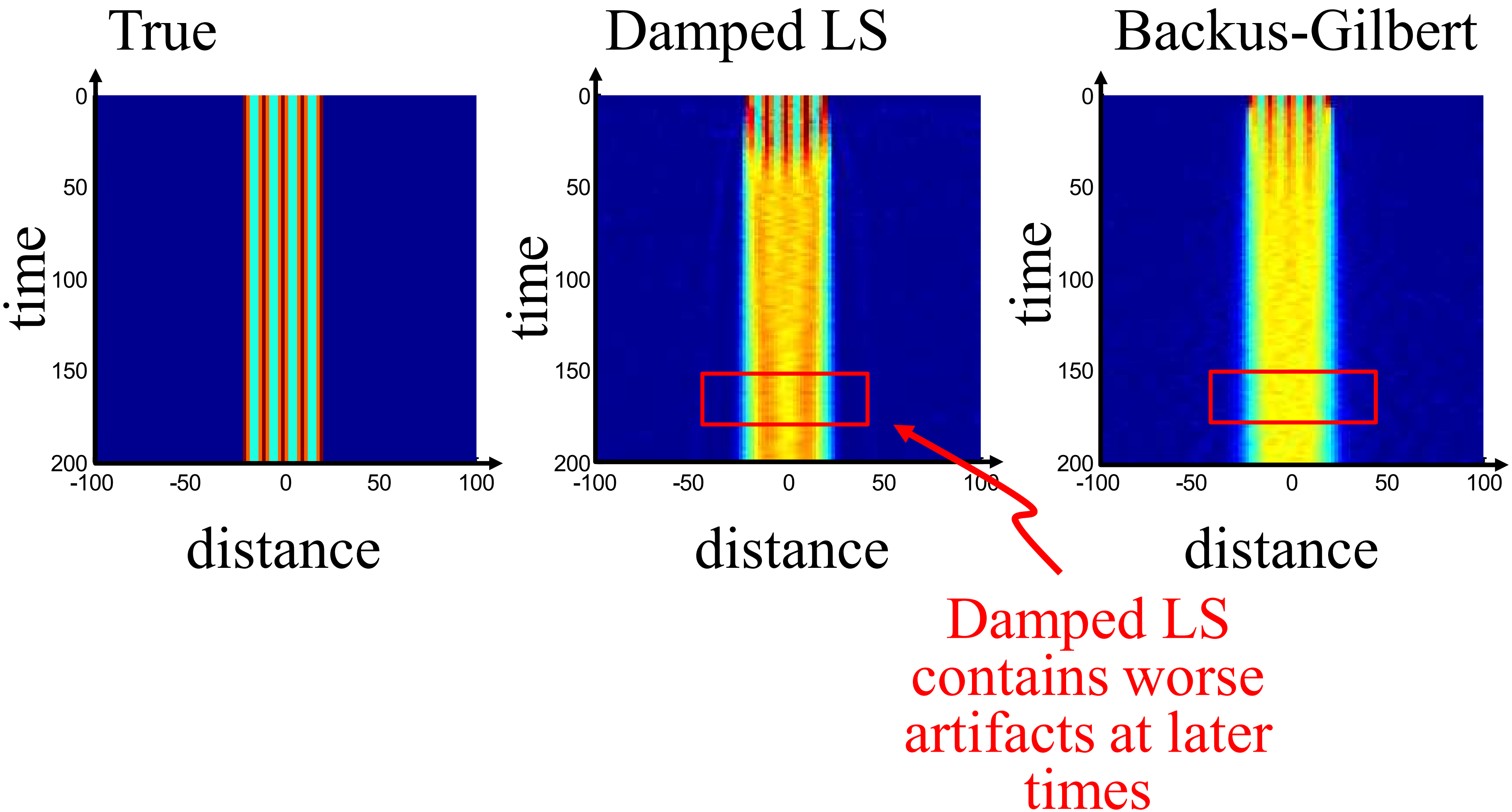
estimated initial temperature distribution as a function of the time of observation



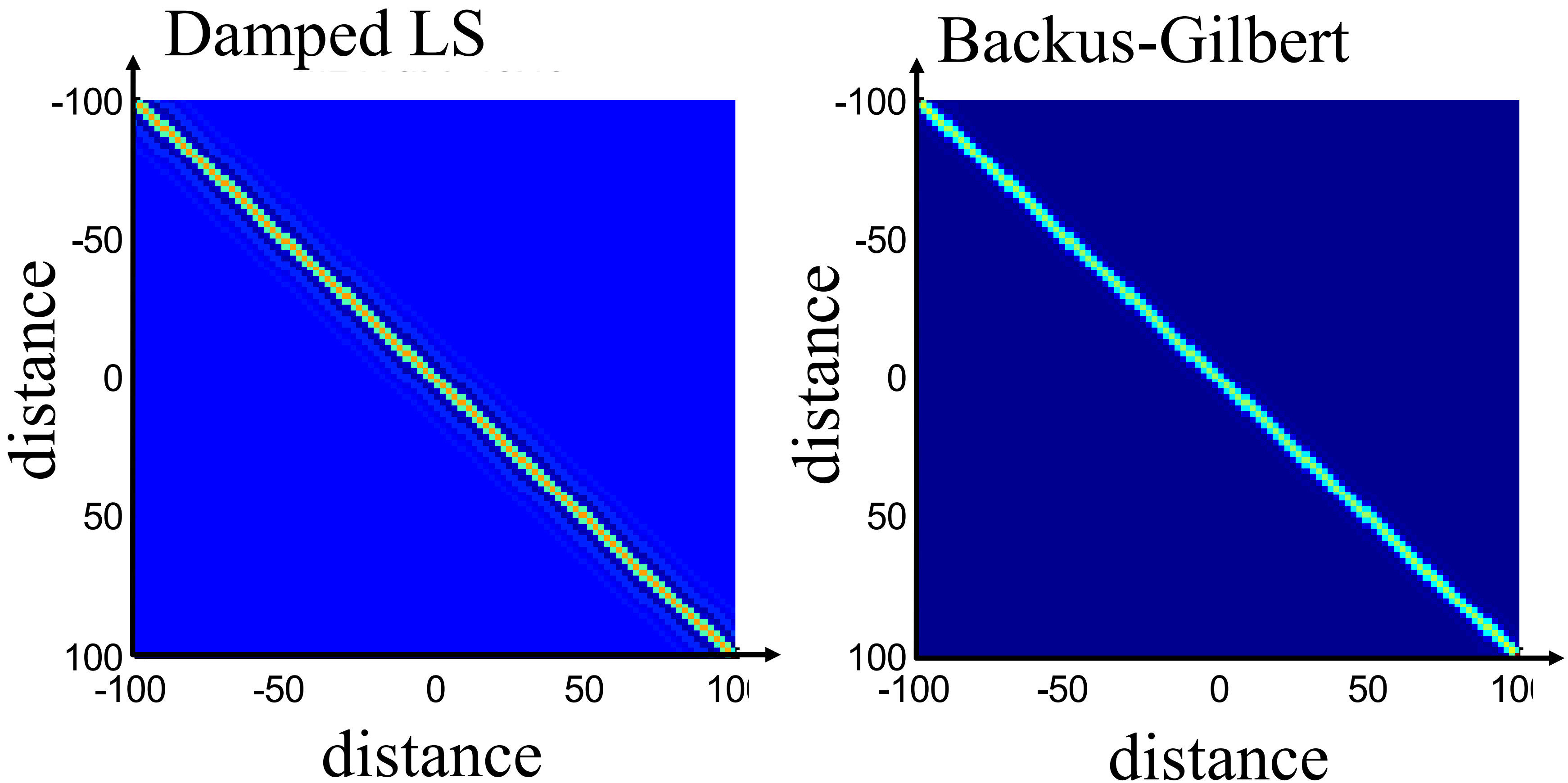
estimated initial temperature distribution as a function of the time of observation



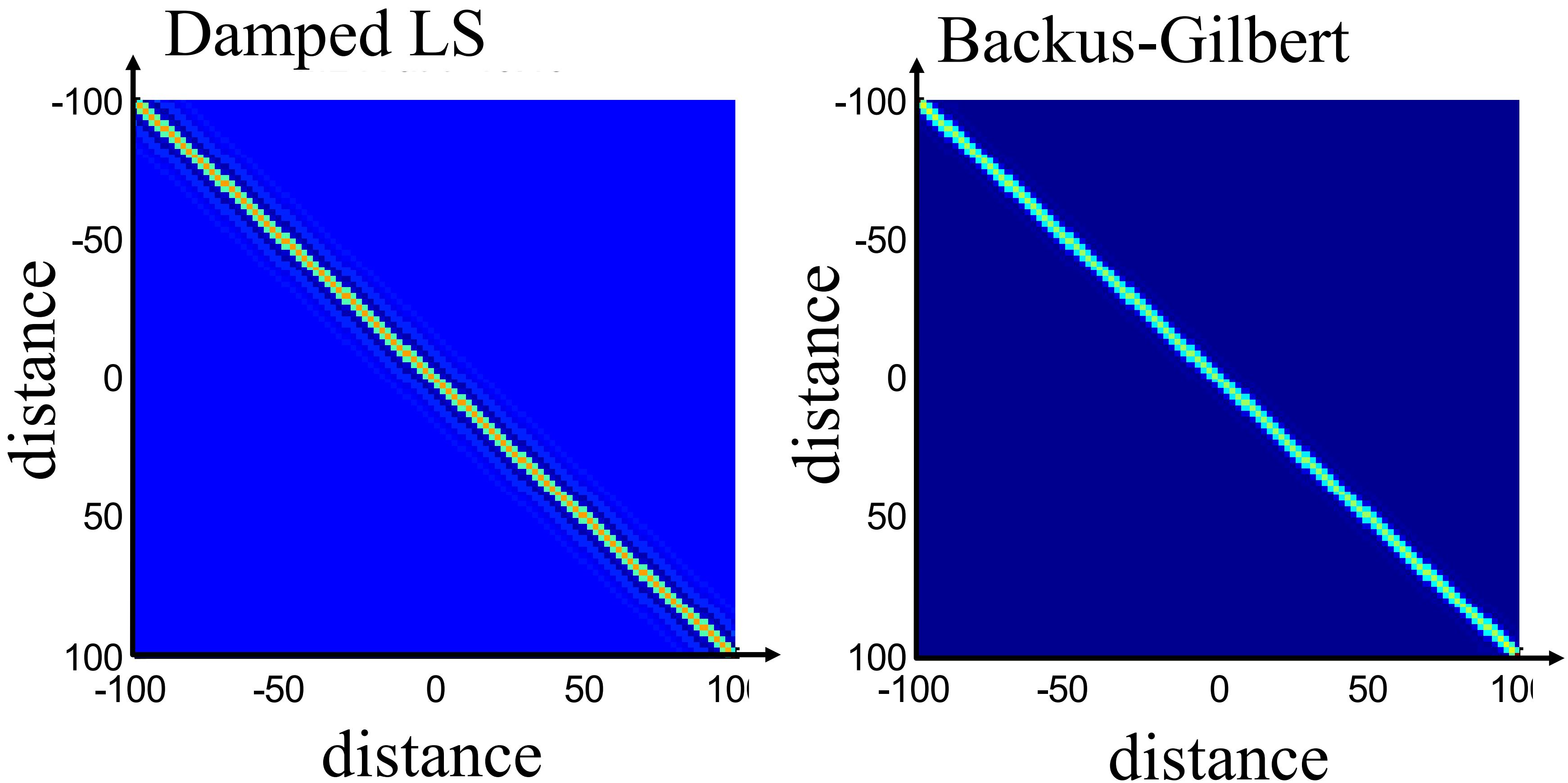
estimated initial temperature distribution as a function of the time of observation



model resolution matrix when for data collected at t=10

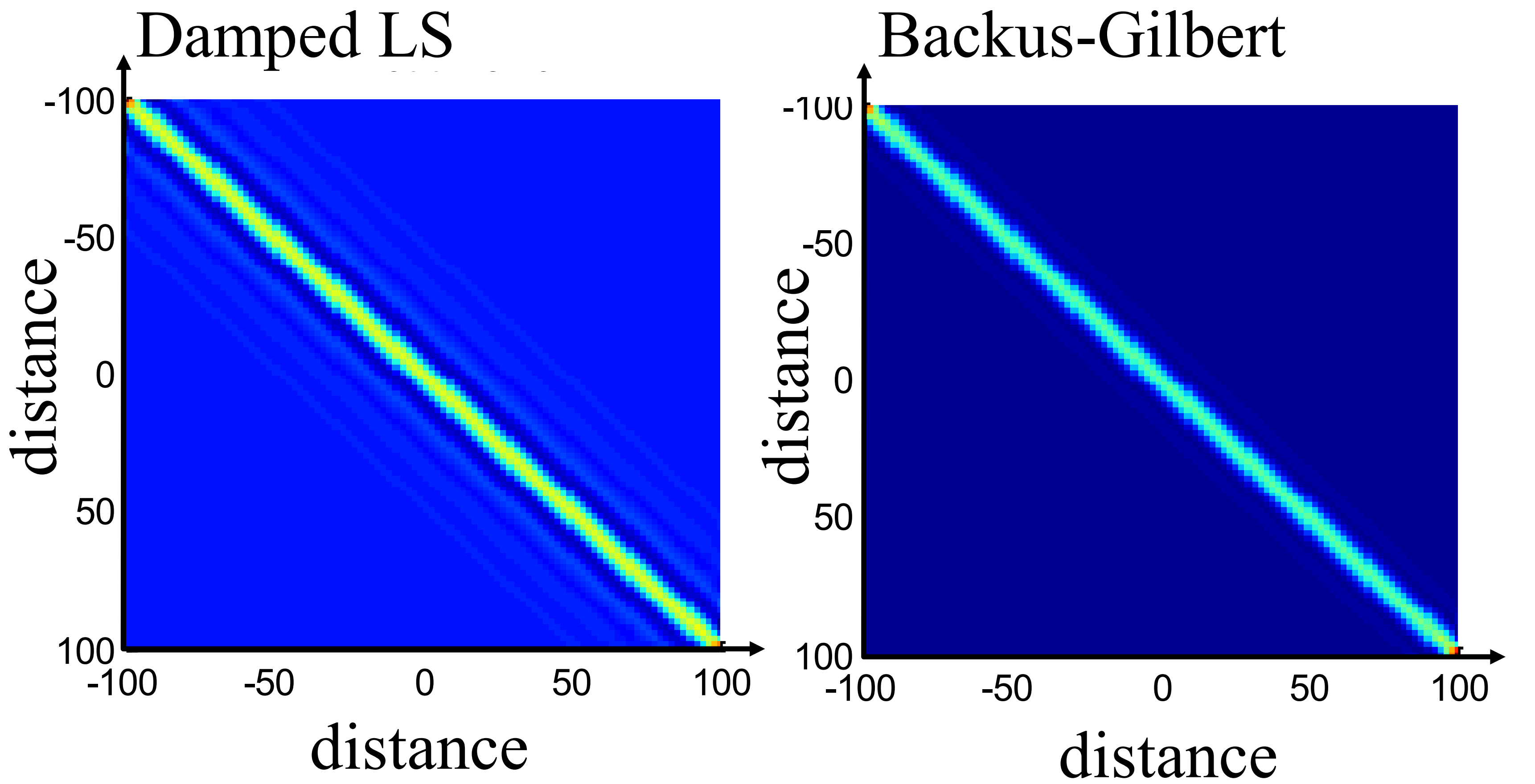


model resolution matrix when for data collected at t=10

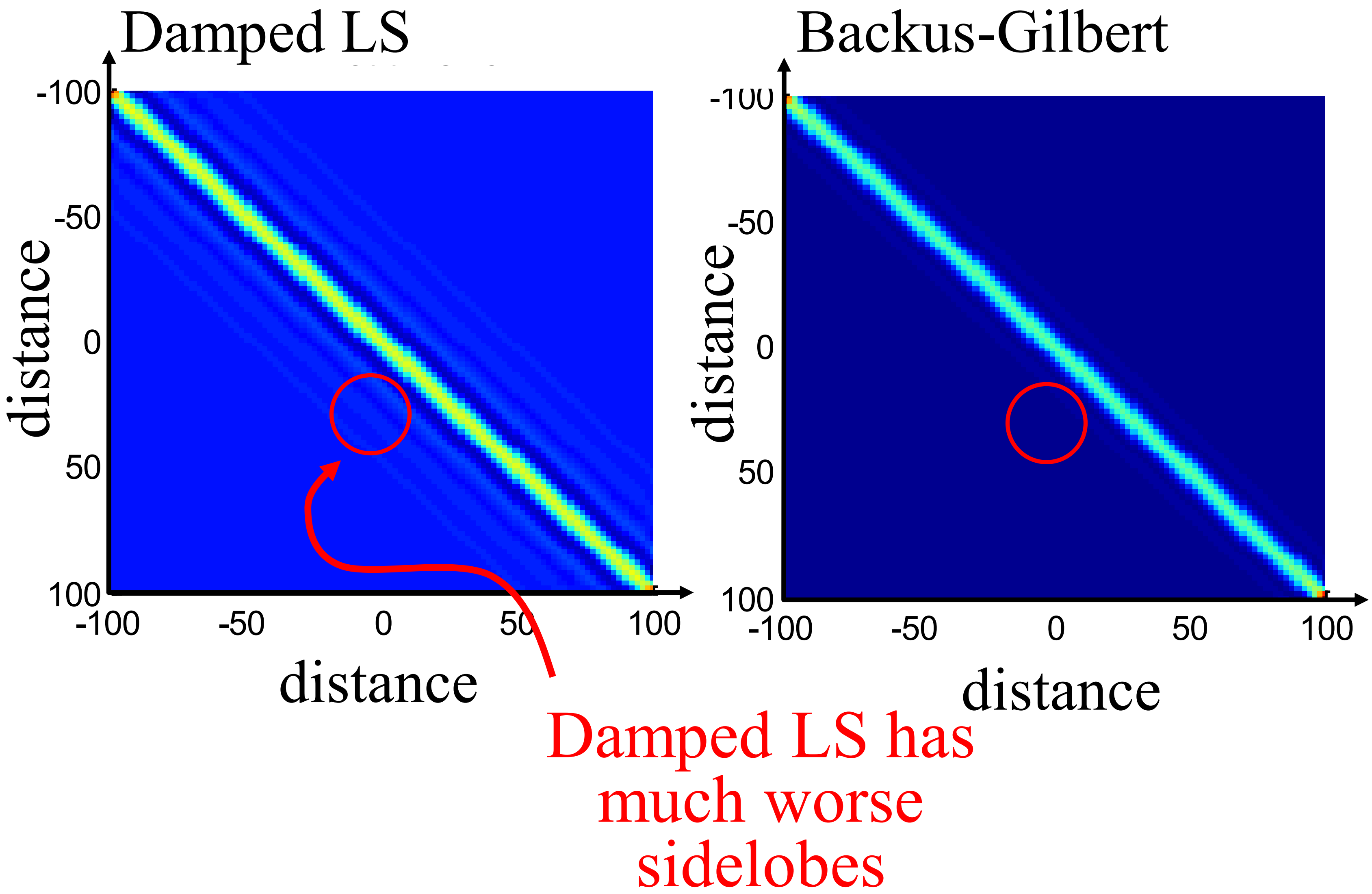


resolution
is similar

model resolution matrix when for data collected at $t=40$



model resolution matrix when for data collected at t=40

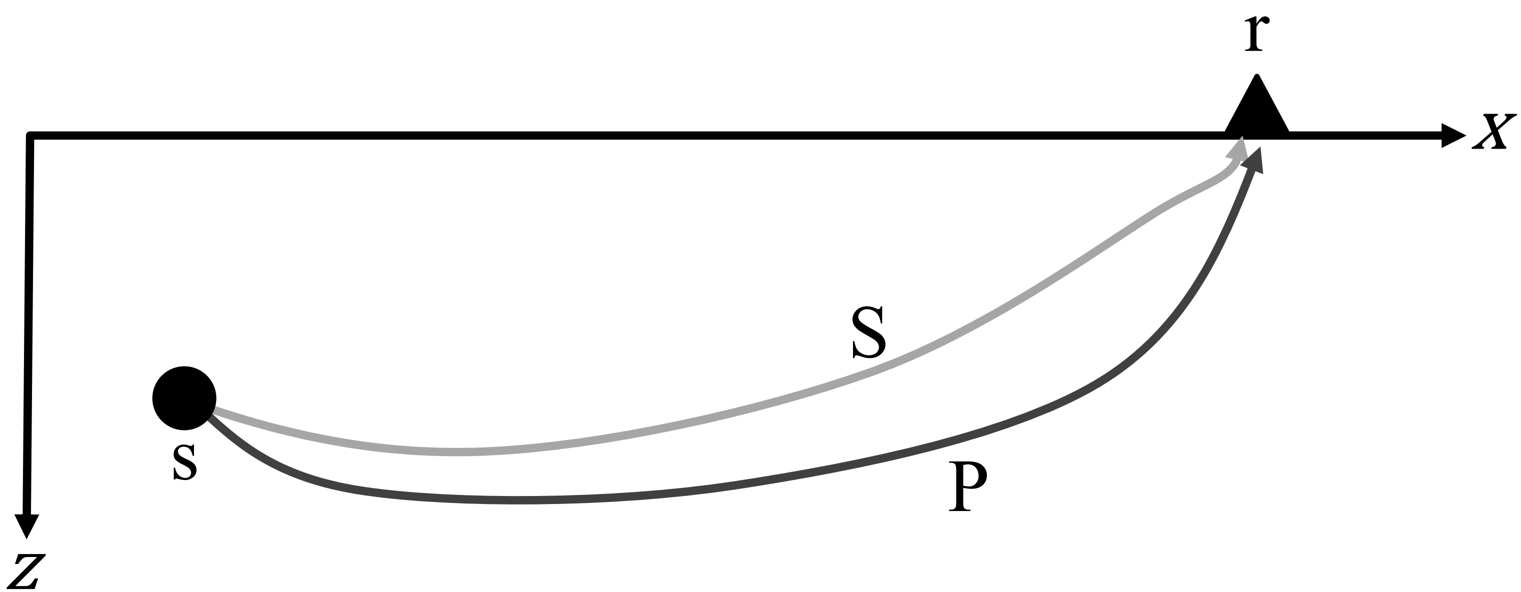


Part 2

earthquake location

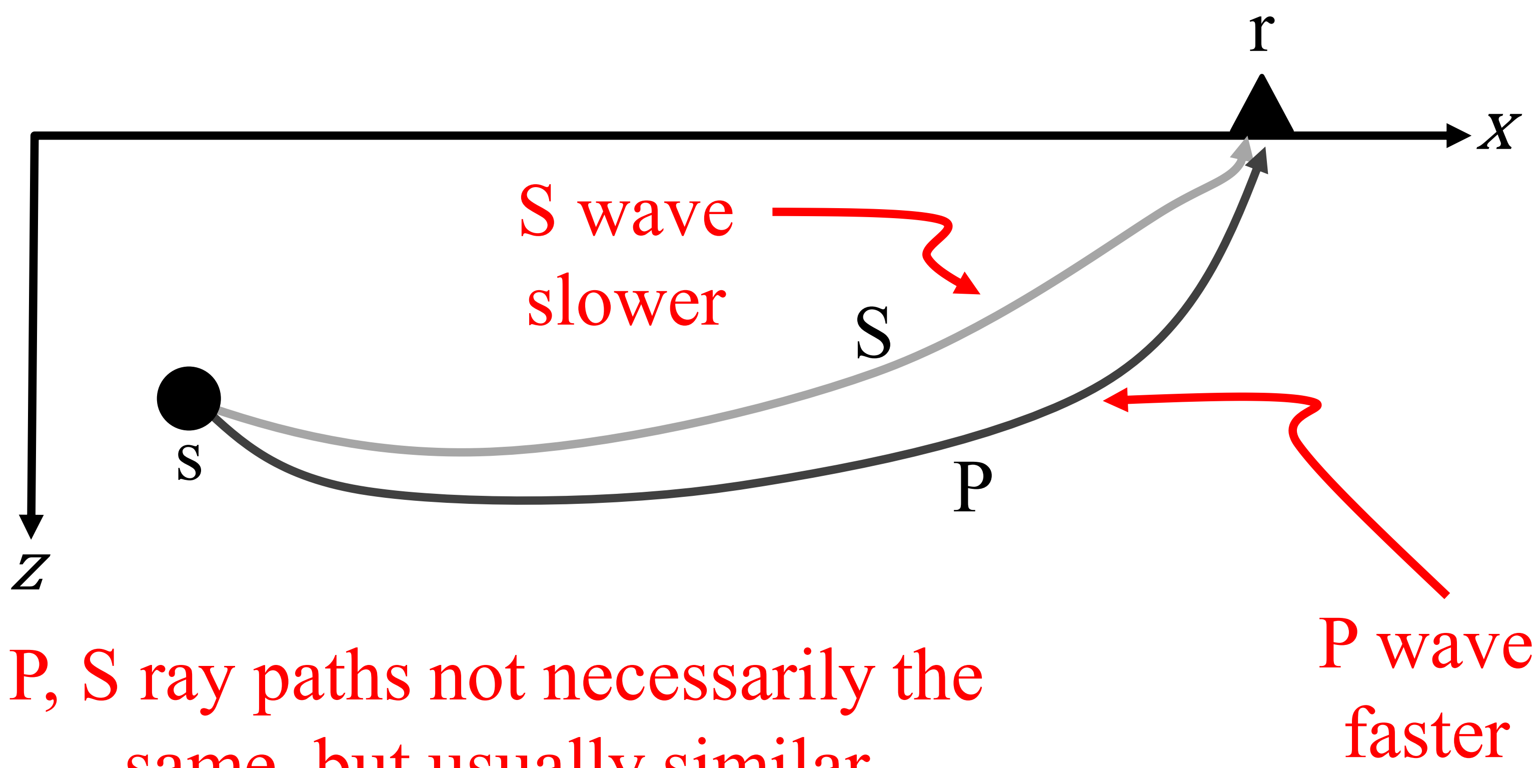
ray approximation

vibrations travel from source to receiver along curved rays



ray approximation

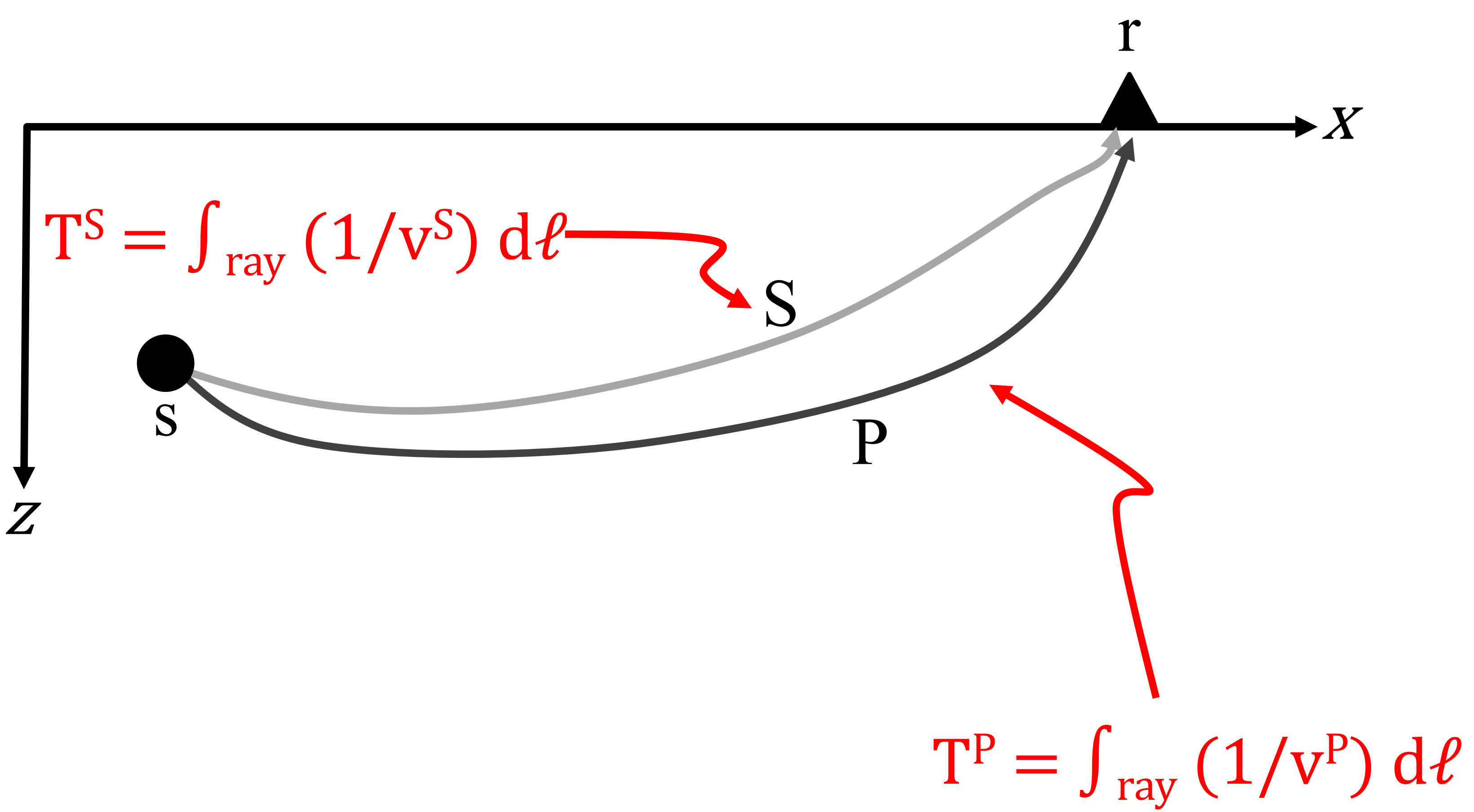
vibrations travel from source to receiver along curved rays



P, S ray paths not necessarily the same, but usually similar

travel time T

integral of slowness along ray path



arrival time = travel time along ray + origin time

$$t_i^P = T_i^P\left(\mathbf{x}^{(0)}, \mathbf{x}^{(\mathrm{i})}\right) + t_0 \quad \text{and} \quad t_i^S = T_i^S\left(\mathbf{x}^{(0)}, \mathbf{x}^{(\mathrm{i})}\right) + t_0$$

arrival time = travel time along ray + origin time

$$t_i^P = T_i^P\left(\mathbf{x}^{(0)}, \mathbf{x}^{(\mathrm{i})}\right) + t_0 \quad \text{and} \quad t_i^S = T_i^S\left(\mathbf{x}^{(0)}, \mathbf{x}^{(\mathrm{i})}\right) + t_0$$

data

data

earthquake location 3 model parameters

earthquake origin time 1 model parameter

arrival time = travel time along ray + origin time

$$t_i^P = T_i^P\left(\mathbf{x}^{(0)}, \mathbf{x}^{(\mathrm{i})}\right) + t_0 \quad \text{and} \quad t_i^S = T_i^S\left(\mathbf{x}^{(0)}, \mathbf{x}^{(\mathrm{i})}\right) + t_0$$

explicit nonlinear equation

4 model parameters
up to 2 data per station

arrival time = travel time along ray + origin time

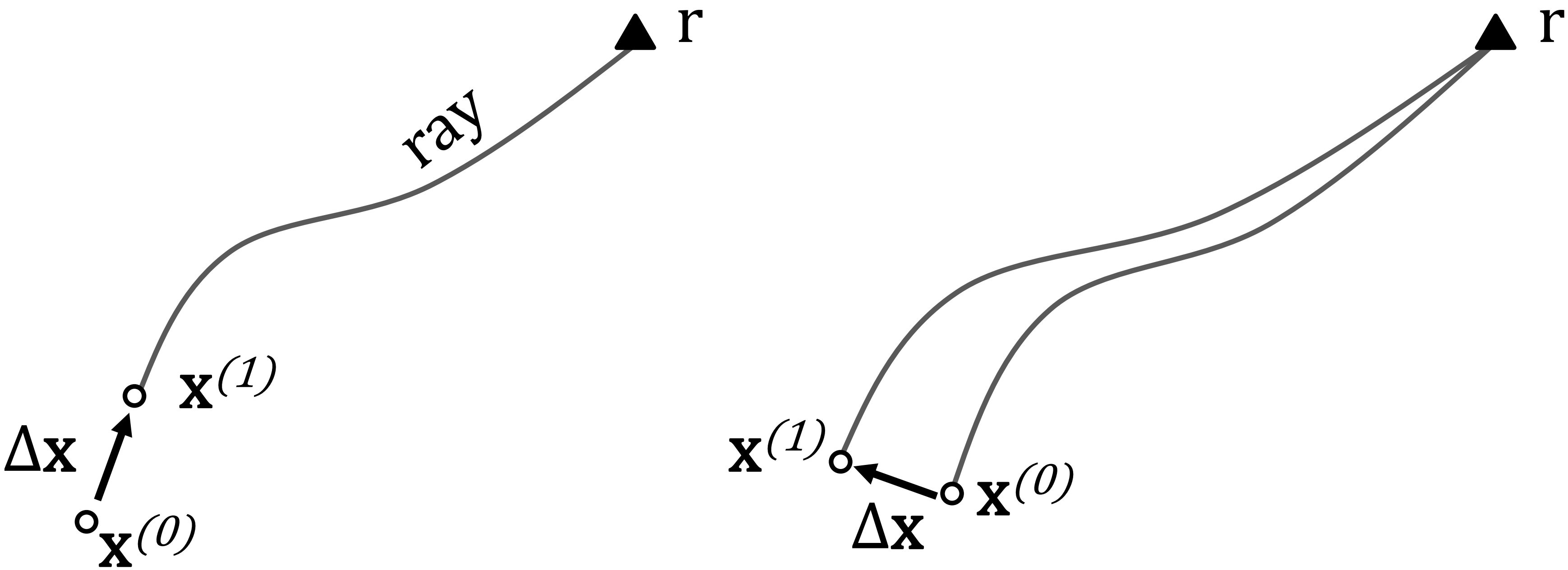
$$t_i^P = T_i^P\left(\mathbf{x}^{(0)}, \mathbf{x}^{(\mathrm{i})}\right) + t_0$$

linearize around trial source location $\mathbf{x}^{(\mathrm{p})}$

$$\mathrm{t_i}^{\mathrm{P}} = \mathrm{T_i}^{\mathrm{P}}(\mathbf{x}^{(\mathrm{p})},\mathbf{x}^{(\mathrm{i})}) + [\nabla \mathrm{T_i}^{\mathrm{P}}] \bullet \Delta \mathrm{x} + \mathrm{t_0}$$

trick is computing this gradient

Geiger's principle



$$[\nabla T_i^P] = -\mathbf{s}/v$$

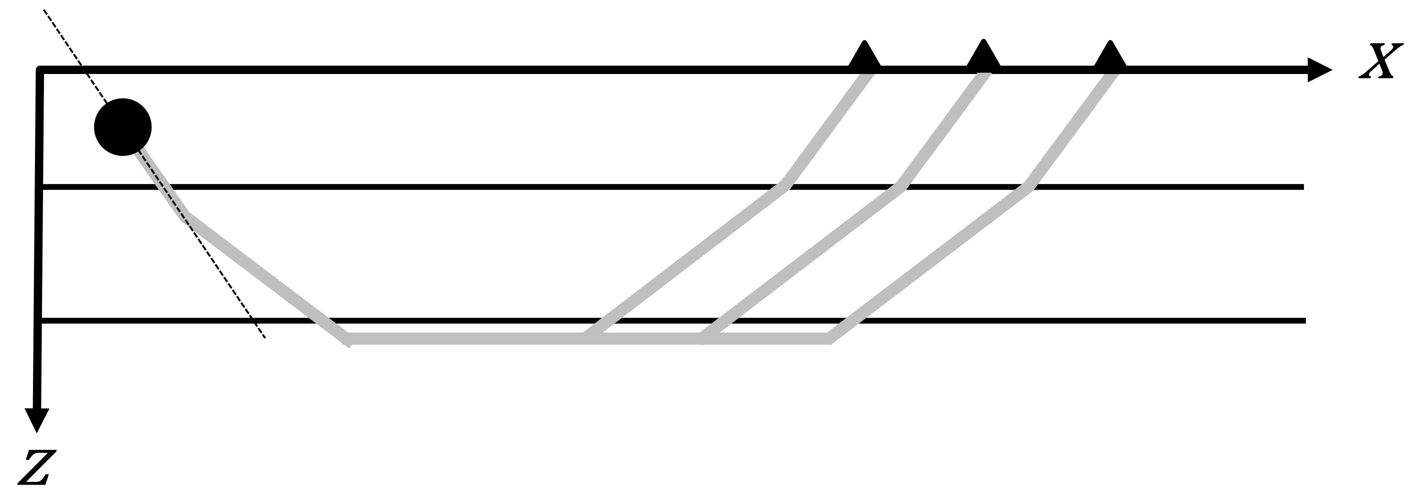
unit vector parallel to ray pointing away from receiver

linearized equation

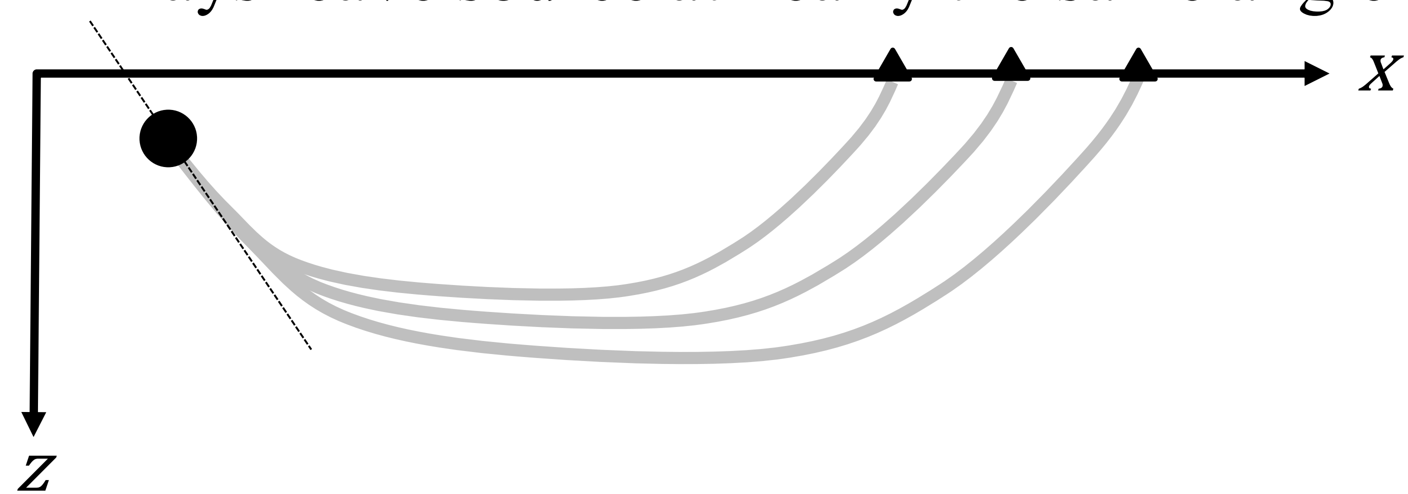
$$\begin{bmatrix} -s_1^{(1)P}/v_0^P & -s_2^{(1)P}/v_0^P & -s_3^{(1)P}/v_0^P & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -s_1^{(N)P}/v_0^P & -s_2^{(N)P}/v_0^P & -s_3^{(N)P}/v_0^P & 1 \\ -s_1^{(1)S}/v_0^S & -s_2^{(1)S}/v_0^S & -s_3^{(1)S}/v_0^S & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -s_1^{(N)S}/v_0^S & -s_2^{(N)S}/v_0^S & -s_2^{(N)S}/v_0^S & 1 \end{bmatrix} \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \\ \Delta z_0 \\ t_0 \end{bmatrix} = \begin{bmatrix} t_1^P - T_1^P \\ \cdots \\ t_N^P - T_N^P \\ t_1^S - T_1^S \\ \cdots \\ t_N^S - T_N^S \end{bmatrix}$$

Common circumstances when earthquake far from stations

All rays leave source at the same angle



All rays leave source at nearly the same angle



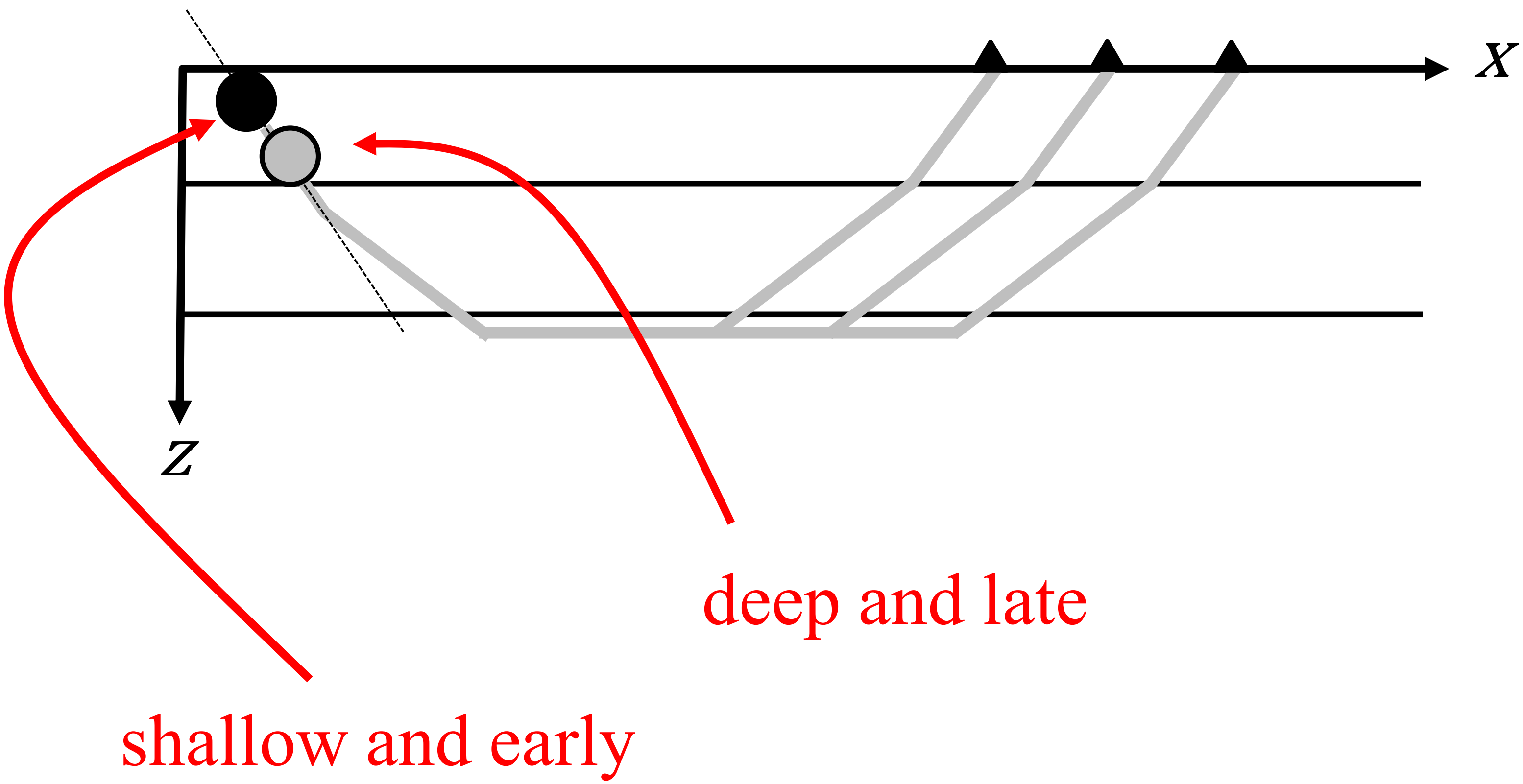
then, if only P wave data is available

$$\begin{bmatrix} -s_1^{(1)P}/v_0^P & -s_2^{(1)P}/v_0^P & -s_3^{(1)P}/v_0^P & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -s_1^{(N)P}/v_0^P & -s_2^{(N)P}/v_0^P & -s_3^{(N)P}/v_0^P & 1 \\ . & & & \\ & & & . \\ . & & & \end{bmatrix} \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \\ \Delta z_0 \\ t_0 \end{bmatrix} = \begin{bmatrix} t_1^P - T_1^P \\ \cdots \\ t_N^P - T_N^P \\ t_1^S - T_1^S \\ \cdots \\ t_N^S - T_N^S \end{bmatrix}$$

(no S waves)

these two columns are proportional to one-another

depth and origin time trade off

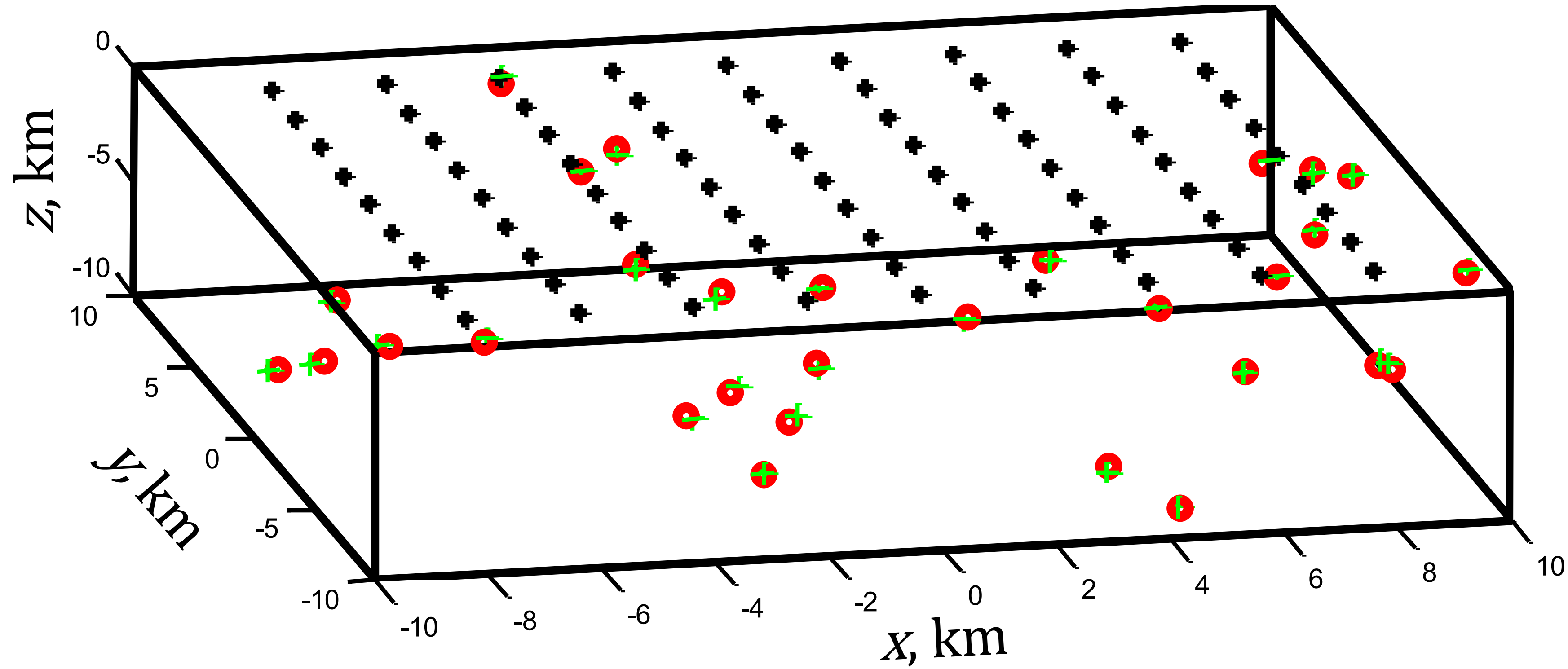


Solution Possibilities

1. Damped Least Squares:
   Matrix **G** is not sparse
   no analytic version of $\mathbf{G}^T\mathbf{G}$ is available
   M=4 is tiny
   experiment with values of ε^2
   `mest=(G'*G+e2*eye(M,M))\(G'*d)`

test case has earthquakes "inside of array"

2. Singular Value Decomposition
   to detect case of depth and origin time trading off

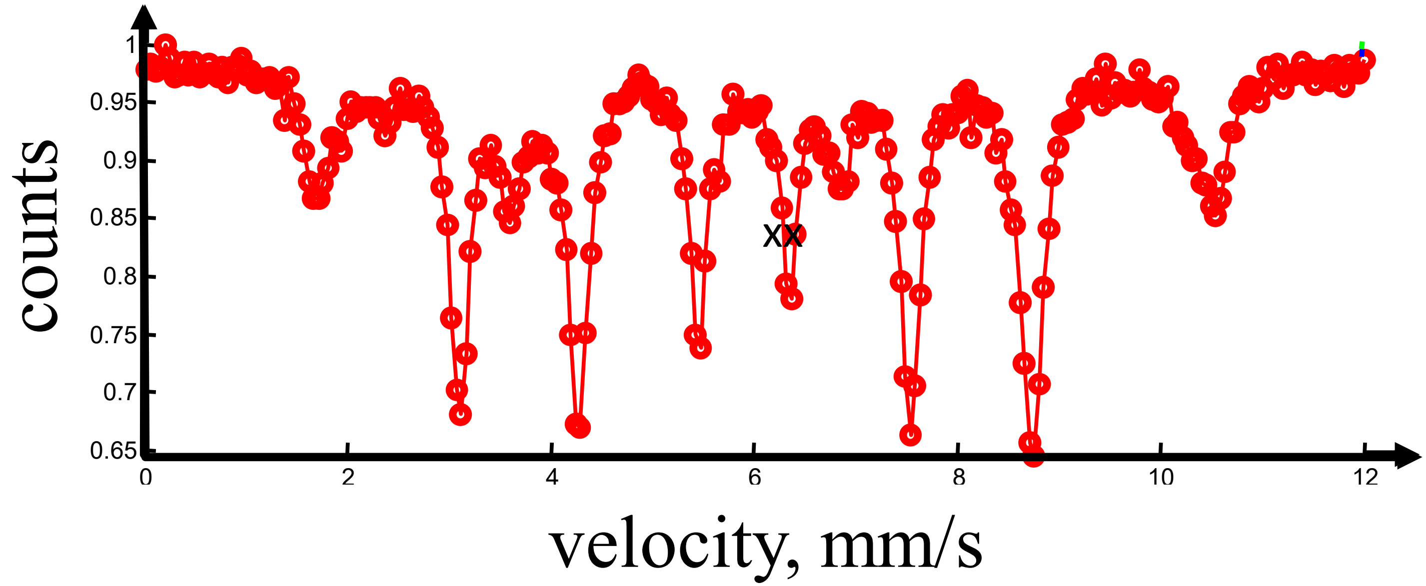


0
-5
-10
z, km
10
5
0
-5
-10
y, km
-10
-8
-6
-4
-2
0
2
4
6
8
10
x, km

Part 3

fitting of spectral peaks

typical spectrum consisting of overlapping peaks



what shape are the peaks?

Gaussian: $$d_i = \sum_{j=1}^{q} \frac{A_j}{(2\pi)^{½} c_j} \exp\left[-\frac{\left(z_i - f_j\right)^2}{2c_j^2}\right]$$

Lorentzian: $$d_i = \sum_{j=1}^{q} \frac{A_j c_j^2}{\left(z_i - f_j\right)^2 + c_j^2}$$

what shape are the peaks?

Gaussian: $$d_i = \sum_{j=1}^{q} \frac{A_j}{(2\pi)^{1/2} c_j} \exp\left[-\frac{\left(z_i - f_j\right)^2}{2c_j^2}\right]$$

Lorentzian: $$d_i = \sum_{j=1}^{q} \frac{A_j c_j^2}{\left(z_i - f_j\right)^2 + c_j^2}$$

try both

use *F* test to test whether one is better than the other

what shape are the peaks?

3 unknowns per peak

data

Gaussian: $$d_i = \sum_{j=1}^{q} \frac{A_j}{(2\pi)^{1/2} c_j} \exp\left[-\frac{(z_i - f_j)^2}{2c_j^2}\right]$$

data

Lorentzian: $$d_i = \sum_{j=1}^{q} \frac{A_j c_j^2}{(z_i - f_j)^2 + c_j^2}$$

3 unknowns per peak

both cases:
explicit nonlinear problem

linearize using analytic gradient

Gaussian:

$$\partial g_i/\partial A_j = \left[1/(2\pi)^{1/2} c_j\right] \exp\left[-(z_i - f_j)^2/2c_j^2\right]$$

$$\partial g_i/\partial f_j = \left[\frac{A_j}{(2\pi)^{1/2} c_j}\right]\left[(z_i - f_j)/c_j^2\right] \exp\left[-(z_i - f_j)^2/2c_j^2\right]$$

$$\partial g_i/\partial c_j = \left[\frac{A_j}{(2\pi)^{1/2} c_j^2}\right]\left[\left((z_i - f_j)^2/c_j^2\right) - 1\right] \exp\left[-(z_i - f_j)^2/2c_j^2\right]$$

linearize using analytic gradient

Lorentzian:

$$\partial g_i/\partial A_j = c_j^2/\left[(z_i - f_j)^2 + c_j^2\right]$$

$$\partial g_i/\partial f_j = 2A_j c_j^2 (z_i - f_j)/\left[(z_i - f_j)^2 + c_j^2\right]^2$$

$$\partial g_i/\partial c_j = 2A_j c_j/\left[(z_i - f_j)^2 + c_j^2\right] - 2A_j c_j^3/\left[(z_i - f_j)^2 + c_j^2\right]^2$$

issues

how to determine

number q of peaks

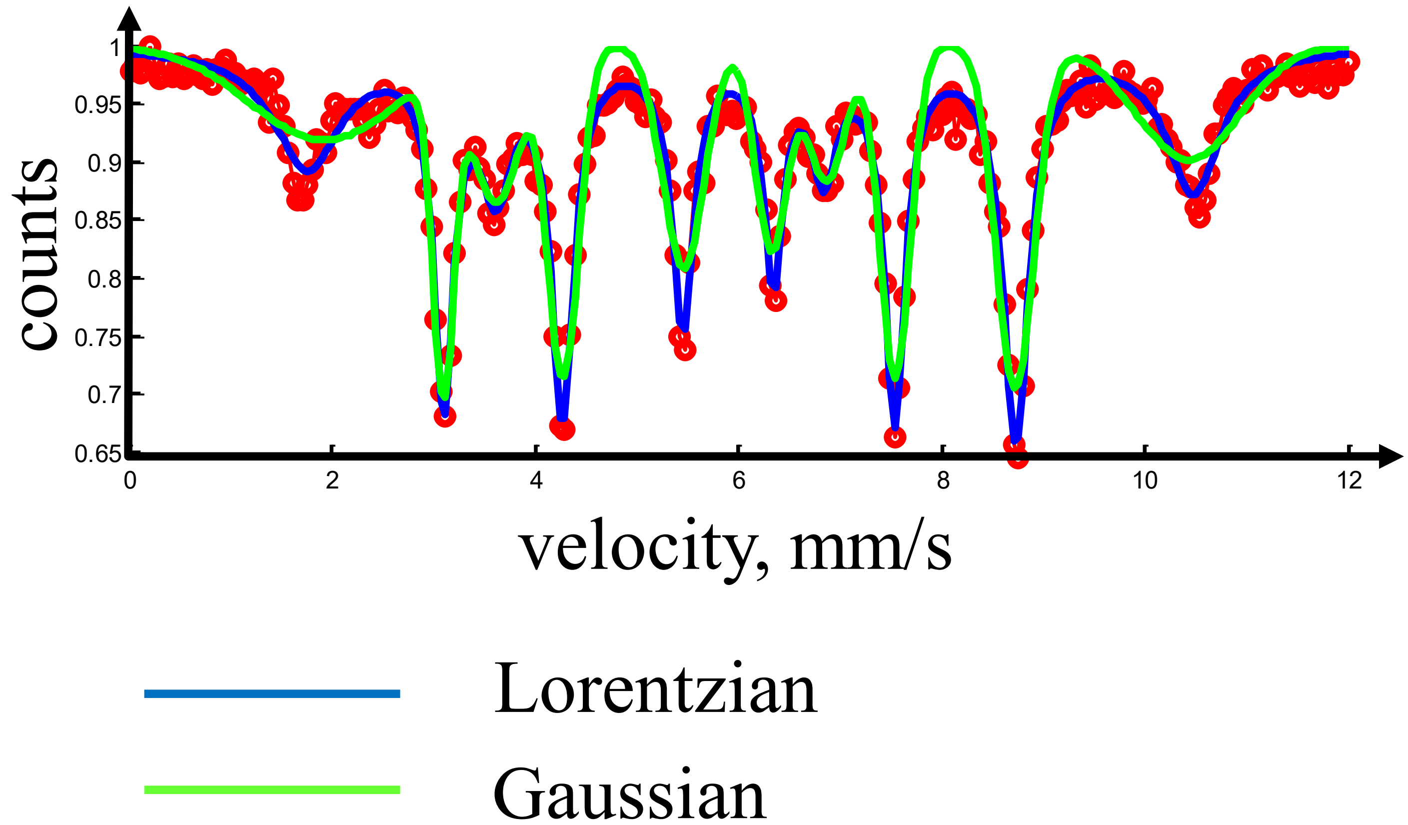
trial A_i c_i f_i of each peak

our solution

have operator click mouse
computer screen
to indicate position of each peak

MatLab code for graphical input

```
K=0;
for k = [1:20]
    p = ginput(1);
    if( p(1) < 0 )
        break;
    end
    K=K+1;
    a(K) = p(2)-A;
    v0(K)=p(1);
    c(K)=0.1;
end
```


counts
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
2
4
6
8
10
12
velocity, mm/s
Lorentzian
Gaussian

Results of F test

```
Fest = E_normal/E_lorentzian: 4.230859
P(F<=1/Fest||F>=Fest) = 0.000000
```

Lorentzian better fit
to 99.9999% certainty