MRN009

Perturbation Theory for eigenvalue-eigenvector problems, from wilkenson

J H wilkenson - The algebraic eigenvalue problem - clarendon press - oxford, 1965, pp 66-70, pp 93-102

given Matrix A such that

$$A x_i = \lambda_i x_i$$

$$y_i A = y_i \lambda_i$$

x_i, y_i orthonormalized

then consider $C = (A + \epsilon B)$; B arbitrary matrix

for sufficiently small ϵ then eigenvalues, eigenvector of $C(\epsilon)$ are:

$$\lambda_i(\epsilon) = \lambda_i + k_i \epsilon \qquad x_i(\epsilon) = x_i + \epsilon z_i$$

define $\quad s_i = y_i^T x_i \qquad \beta_{ij} = y_i^T B x_j$

Then

$$k_i = \beta_{ii} / s_i$$

$$z_i = \sum_{j \neq i} \frac{\beta_{ji} x_j}{(\lambda_i - \lambda_j) s_j}$$

for symmetric matrices A ; B $\quad x_i = y_i \quad s_i = 1 \quad \beta_{ij} = x_i^T B x_j$

$$k_i = \beta_{ii} \qquad z_i = \sum_{j \neq i} \frac{\beta_{ji} x_j}{(\lambda_i - \lambda_j) s_j}$$

now in our case we have initial matrix

B with eigenvalues λ_i and eigenvectors $f^{(i)} = (p_i, q_i, r_i)$

then the first eigenvector of $B + \delta B$ is given by

$$f_i^{(1)} \approx {}^{\circ}f_i^{(1)} + \frac{f_i^{(2)}}{(\lambda_1-\lambda_2)} \beta_{21} + \frac{f_i^{(2)}}{(\lambda_1-\lambda_3)} \beta_{31}$$

now $\beta_{ij} = f_i^T \delta B_{ij} f_j$

$$= (p_i p_j)\delta B_{11} + (p_i q_j + q_i p_j)\delta B_{12} + (p_i r_j + r_i p_j)\delta B_{13} + (q_i q_j)\delta B_{22} + (q_i r_j + r_i q_j)\delta B_{23} + (r_i r_j)\delta B_{33}$$

$$= \sum_k \alpha_k(i,j)\,\delta b_k \quad ; \quad b = [B_{11}, B_{12}, B_{13}, B_{22}, B_{23}, B_{33}]$$

so

$$f_i^{(1)} = {}^{\circ}f_i^{(1)} + \frac{f_i^{(2)}}{(\lambda_1-\lambda_2)} \sum_k \alpha_k(2,1)\,\delta b_k + \frac{f_i^{(3)}}{(\lambda_1-\lambda_3)} \sum_k \alpha_k(3,1)\,\delta b_k$$

$$= {}^{\circ}f_i^{(1)} + \sum_k \left\{ \frac{f_i^{(2)}}{(\lambda_1-\lambda_2)} \alpha_k(2,1) + \frac{f_i^{(3)}}{(\lambda_1-\lambda_3)} \alpha_k(3,1) \right\} \delta b_k$$

$$= {}^{\circ}f_1^{(1)} + \sum_{k=1}^{6} \xi_{ik}\,\delta b_k$$