Parseval's Thm.

Define fourier transform $f(\omega) = \int_t f(t)\, e^{\pm i\omega t}\, dt$

inverse f. t. $f(t) = \frac{1}{2\pi} \int_\omega f(\omega)\, e^{\mp i\omega t}\, dt$

Note $\int_\omega e^{i\omega(t-t')}\, d\omega = 2\pi\, \delta(t-t')$

because $2\pi \int_t \delta(t-t')\, e^{-i\omega t}\, dt = 2\pi\, e^{-i\omega t'}$ is a fourier tr.

so $\frac{1}{2\pi} \int_\omega 2\pi\, e^{-i\omega t'} e^{i\omega t}\, d\omega = \int_\omega e^{i\omega(t-t')}\, d\omega$ is its inv. tr.

Parseval's Thm $\int_t f^2(t)\, dt = \frac{1}{2\pi} \int_\omega |f(\omega)|^2\, d\omega$

proof. $\int_\omega |f(\omega)|^2 d\omega = \int_\omega f(\omega)\, f^*(\omega)\, d\omega =$

$\int_\omega \int_t f(t)\, e^{i\omega t}\, dt \int_{t'} f(t')\, e^{-i\omega t'}\, dt'\, d\omega =$

$\int_t f(t) \int_{t'} f(t') \int_\omega e^{i\omega(t-t')}\, d\omega\, dt'\, dt =$

$\int_t f(t) \int_{t'} f(t')\, 2\pi\, \delta(t-t')\, dt'\, dt =$

$2\pi \int_t f(t)\, f(t)\, dt = 2\pi \int_t f^2(t)\, dt$

Discrete version.

Define Discrete fourier transform $\tilde{f}(\omega_j) = \sum_i f(t_i)\, e^{i\omega_j t_i}$

Then $f(\omega_i) \approx \Delta t\, \tilde{f}(\omega_i)$

also $f_{ny} = \frac{1}{2\Delta t} = \frac{N}{2} \Delta f = \frac{N}{2} \frac{\Delta\omega}{2\pi}$ so $\Delta t\, \Delta\omega = \frac{2\pi}{N}$

$$\int_t f^2(t)\,dt = \frac{1}{2\pi}\int_\omega |f(\omega)|^2\,d\omega$$

$$\Delta t \sum_i f^2(t_i) = \frac{1}{2\pi}\Delta\omega \sum_i |f(\omega_i)| = \frac{1}{2\pi}\Delta\omega(\Delta t)^2 \sum_i |\tilde{f}(\omega_i)|^2$$

$$\sum_i f^2(t_i) = \frac{1}{2\pi}\Delta\omega\,\Delta t \sum_i |\tilde{f}(\omega_i)|^2 = \frac{1}{N}\sum_i |\tilde{f}(\omega_i)|^2$$

example

t_0	t_1	t_2	t_3
1	0	0	0

$\xrightarrow{DFT}$

ω_0	ω_1	ω_2	ω_{-1}
1	1	1	1

$N=4$

$$\sum_i f^2(t_i) = 1$$

$$\sum_i f^2(\omega_i) = 4$$

$$\frac{1}{N}\sum f^2(\omega_i) = 1$$