Consider the principle:

"The wave front advances normal to itself at a speed given by the local medium velocity, c."

If T gives the traveltime, then the above statement is equivalent to $\boxed{s\underline{\ell} = \nabla T}$. Here $s = 1/c$ and $\underline{\ell}$ is a unit vector normal to the wavefront.

The Eikonal equation is just $s\underline{\ell} = \nabla T$ dotted with itself:

$$\boxed{s^2 = \nabla T \cdot \nabla T}$$

The ray equation is an equation involving only s and $\underline{\ell}$, not T. The vector field $\underline{\ell}$ cannot be specified arbitrarily, since it must ultimately be related to ∇T by $s\underline{\ell} = \nabla T$. Thus $s\underline{\ell}$ is a <u>conservative vector field</u> and has no curl:

$$\varepsilon_{ijk}\,(s\ell_k)_{,j} = 0$$

$$\varepsilon_{ijk}\, s_{,j}\, \ell_k + \varepsilon_{ijk}\, s\, \ell_{k,j} = 0$$

$$s\, \nabla \times \underline{\ell} = \underline{\ell} \times \nabla s$$

$$\boxed{\nabla \times \underline{\ell} = \underline{\ell} \times \frac{1}{s} \nabla s}$$

(for a ray $\underline{x}(s)$)

The ray eqn is derived by taking $\underline{l}\times$ the previous equation:

$$\underline{l}\times(\nabla\times\underline{l}) = \underline{l}\times(\underline{l}\times\tfrac{1}{s}\nabla s)$$

note $[\ell\times(\nabla\times\ell)]_i =$

$\varepsilon_{ijk}\, l_j\, \varepsilon_{klm}\, l_{l,m} =$

$\varepsilon_{ijk}\,\varepsilon_{klm}\, l_j\, l_{l,m} =$

$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, l_j\, l_{l,m} =$

$l_m\, l_{i,m} - l_m\, l_{m,i} =$

$[\nabla l\cdot l - l\cdot\nabla l]_i$

note $\nabla l\cdot l = l_m l_{i,m} = \dfrac{dx_m}{ds}\dfrac{dl_i}{dx_m} = \dfrac{dl_i}{ds}$

where s = arclength along ray

note $l\cdot\nabla l = 0$ since $\underline{l}\perp\nabla\underline{l}$ follows from $\underline{l}$ being a unit vector:

$$0 = (l_j l_j)_{,i} = 2\, l_{j,i}\, l_j = 2[\underline{l}\cdot\nabla\underline{l}]_i$$

so ray eqn is

$$\boxed{\frac{d\underline{l}}{ds} = \underline{l}\times(\underline{l}\times\tfrac{1}{s}\nabla s)}$$