MRN046 Jiongping! XEROX 3 for students

Question Suppose we know $\underline{\underline{T}}(\underline{x})$ everywhere in an elastic solid. How do we find $\underline{u}(\underline{x})$?

Solution $\underline{\underline{T}}(x)$ implies $\underline{\underline{\varepsilon}}(\underline{x})$, so we have the following equations for $\underline{u}$: $u_{i,j} + u_{j,i} = 2\varepsilon_{ij}$.

Let's examine these in 2-D. They are

$$\frac{\partial u_x}{\partial x} = \varepsilon_{xx}, \quad \frac{\partial u_y}{\partial y} = \varepsilon_{yy}, \quad \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 2\varepsilon_{xy}$$

These are essentially equations for the gradient of $\underline{u}$. But one equation is missing, the rotation equation,

$$\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} = 2\Omega_{xy}$$

If we knew it, then we could solve for the gradient

$$\nabla \underline{u} = \begin{bmatrix} \partial u_x/\partial x & \partial u_x/\partial y \\ \partial u_y/\partial x & \partial u_y/\partial y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & (\varepsilon_{xy} + \Omega_{xy}) \\ (\varepsilon_{xy} - \Omega_{xy}) & \varepsilon_{yy} \end{bmatrix}$$

Then we could use the fundamental theorem of calculus

$$u(\underline{x}) = u_0 + \int_{\substack{\text{any path from} \\ x_0 \text{ to } \underline{x}}} \hat{t} \cdot \nabla u \, ds$$

Here $u_0 = u(x_0)$ is an integration constant and $\hat{t}$ is the tangent to the path. So the stress (or strain) alone is insufficient to specify the displacement. You need the rotation everywhere ($\underline{\Omega}(x)$) and the displacement at one point (the integration constant, $u(x=0)$). Note that $\underline{\Omega}(x)$ can't be arbitrarily chosen, since there is a compatibility requirement $2\Omega_{xy,xy} = \varepsilon_{xx,yy} - \varepsilon_{yy,xx}$