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Randy Richardson - Some further thoughts -

consider the problem: minimize $\underline{m}^T \underline{\underline{W}}_m \underline{m}$ subject to $\underline{\underline{G}}\,\underline{m} = \underline{d}$.
Generalizing the method in Sec 3.7, and introducing a lagrange multiplier $\underline{\lambda}$, we find we have to solve two equations simultaneously:

$$\underline{\underline{G}}\,\underline{m} = \underline{d} \quad \text{and} \quad \underline{\underline{W}}_m \underline{m} = \underline{\underline{G}}^T \frac{\underline{\lambda}}{2}$$

Now if $\underline{\underline{W}}_m^{-1}$ exists, we can solve for $\underline{m}$ explicitly by first writing $\underline{m} = \underline{\underline{W}}_m^{-1} \underline{\underline{G}}^T \frac{\underline{\lambda}}{2}$, then noting $\underline{d} = \underline{\underline{G}}\,\underline{\underline{W}}_m^{-1} \underline{\underline{G}}^T \frac{\underline{\lambda}}{2}$,
so $\frac{\underline{\lambda}}{2} = [\underline{\underline{G}}\,\underline{\underline{W}}_m^{-1} \underline{\underline{G}}^T]^{-1} \underline{d}$ and finally

$$\underline{m} = \underline{\underline{W}}_m^{-1} \underline{\underline{G}}^T [\underline{\underline{G}}\,\underline{\underline{W}}_m^{-1} \underline{\underline{G}}^T]^{-1} \underline{d}$$

which is essentially 3.38 with the typo corrected. But this presumes that $\underline{W}_m^{-1}$ exists. If it doesn't, then we must write the two equations as:

$$\underline{\underline{G}}\,\underline{m} + 0 \frac{\underline{\lambda}}{2} = \underline{d} \;; \quad \underline{\underline{W}}_m \underline{m} - \underline{\underline{G}}^T \frac{\underline{\lambda}}{2} = 0$$

or as the coupled system

$$\begin{bmatrix} \underline{\underline{W}}_m & -\underline{\underline{G}}^T \\ \underline{\underline{G}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{m} \\ \frac{\underline{\lambda}}{2} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{d} \end{bmatrix}$$

in this case there is no closed form expression for $\underline{m}$, instead $\underline{m}$ and $\frac{\underline{\lambda}}{2}$ must be solved for simultaneously.

Now in the line example, with $d_1 = m_1 x_1 + m_2$,
we have $\underline{\underline{G}} = [x_1 \ \ 1]$, $\underline{D} = [-1 \ \ 1]$, $\underline{\underline{W}}_m = D^T D = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.
$\underline{m}$ is of length 2, and $\underline{\lambda}$ is actually a scalar.
The coupled system is

$$\begin{bmatrix} 1 & -1 & -x_1 \\ -1 & 1 & -1 \\ x_1 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \frac{\lambda_1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}$$

note that the determinant is $0 + x_1 + x_1 + x_1^2 + 1 + 0 = x_1^2 + 2x_1 + 1 = (x_1+1)^2$
which is not identically zero. The special case is when
$x_1 = (-1)$, in which case the constraint $d_1 = m_1 \cdot (-1) + m_2 = -m_1 + m_2$
is the same equation as the norm tries to minimize.

In the linear case, we can solve the coupled system
explicitly (by computing the necessary determinants)

$$m_1 = m_2 = \frac{d_1}{x_1+1}$$

note that this solution satisfies the data

$$d_1 = m_1 x_1 + m_2 = \frac{d_1 x_1}{x_1+1} + \frac{d_1}{x_1+1} = \frac{d_1(x_1+1)}{(x_1+1)} = d_1$$

and also minimizes $m_1 - m_2$. In fact it is minimized
to zero: $m_1 - m_2 = 0$, a special case.

Best Wishes,

Bill Menke

PS. As you can see, the singularity of $\underline{\underline{W}}_m^{-1}$ posed no insurmountable problems!