Ideal gas law

\[ P = \frac{nRT}{V} \]

Planetary Atmospheric density

\[ \rho = \frac{m}{V} \]

n = number of moles, \( m \) = mass/mole Then

\[ g = \frac{m\alpha}{V} = \frac{m}{RT} \frac{1}{p} = \alpha p \]

where \( \alpha = \frac{m}{RT} \) scales pressure to density

Stress equilibrium

\[ T_{ij,j} + f_i = 0 \]

\( f_i = \) force per unit volume

\[ T_{ij} = -p \delta_{ij} \]

in fluid so

\[ -p_{,i} + f_i = 0 \]

That is \( \nabla p = f \)

Gravitation in radially symmetric object

\[ f = g \frac{\hat{r}}{r} \]

\( g_r = \) radial acceleration

\[ \frac{\partial p}{\partial r} = \frac{g_r}{r} = \alpha p g r \]

Massive planet with tenuous atmosphere

\[ g_r = -g_s \frac{r_s^2}{r^2} \]

\( g_s = \) surface acceleration

\( r_s = \) surface radius

\[ \frac{d}{dr} \left( \frac{dP}{r^2} \right) = -g_s r_s^2 \frac{d^2P}{r^2} \] 

\[ \frac{dp}{dr} = -C \frac{r^2}{d^2} \]

where \( C = g_s r_s^2 \)
Particular solution
\[ \frac{dp}{p} = -c e^{-2z} dz \]
\[ \ln p = c e^{-2z} + \ln a \quad a = \text{const} \]
\[ p = a e^{c/2} \]

General solution, not constant solution e.g., so
\[ p = a e^{c/2} + b e^{c/\sqrt{2}} \]

Boundary condition: \[ p(z = L) = p_0 \quad p(z = \infty) = 0 \]
\[ 0 = a e^{c/2} + b e^{c/\sqrt{2}} \quad \text{so} \quad a = -b e^{c/\sqrt{2}} \]
\[ p_0 = -b e^{c/2} e^{c/\sqrt{2}} + b e^{c/\sqrt{2}} \]
\[ = b e^{c/\sqrt{2}} (e^{c/\sqrt{2}} - 1) \]
\[ b = \frac{p_0}{(e^{c/\sqrt{2}} - 1) e^{c/\sqrt{2}}} \]

\[ p = \frac{p_0}{(e^{c/\sqrt{2}} - 1) e^{c/\sqrt{2}}} \]

Now, suppose \( z = \lambda s + z \) with \( |\lambda| < \lambda_s \)
\[ a^{-1} = (\lambda_s + z)^{-1} = \lambda_s^{-1} (1 + \frac{z}{\lambda_s})^{-1} \approx \lambda_s^{-1} \left(1 - \frac{z}{\lambda_s}\right) \]
\[ e^{cz_1} = e^{c(z_1^{-1} - \frac{z}{\lambda_s^2})} \approx e^{c/\sqrt{2}} e^{-c^2/\lambda_s^2} \]
\[ P = \frac{P_0}{e^{\frac{9s}{r^2}} e^{-\frac{9s x^2}{r^2}}} - 1 \]

\[ = \frac{P_0}{e^{\frac{9s x^2}{r^2}}} (e^{-\frac{9s x^2}{r^2}} - 1) \]

\[ = \lim_{r \to \infty} \frac{P_0}{e^{\frac{9s x^2}{r^2}}} \]

\[ P = P_0 e^{-\frac{9s x^2}{r^2}} \]

Now suppose self-excitation of phase of force \( g_r \)

\[ g_r = -\frac{\gamma M(r)}{r^2} \quad \text{where} \quad M(r) = \frac{4\pi}{3} \int_0^r g(r') r'^2 \, dr' \]

So

\[ \frac{1}{d} \frac{dp}{dr} = -\alpha \frac{\gamma}{r^2} \int_0^r g(r') r'^2 \, dr' \]

\[ \frac{1}{P} \frac{dP}{dr} = -\alpha^2 \frac{r}{r^2} \int_0^r p(r') r'^2 \, dr' \]

\[ \frac{2}{\nu} \frac{d\nu}{dr} \ln p = -\alpha^2 r \int_0^r p(r') r'^2 \, dr' \]

\[ \frac{1}{\nu^2} \frac{d\nu^2}{dr} \ln p = -\alpha^2 \frac{\nu p}{r} \]