Distribution of product of two random variables and its variance. Menke Dec 8, 2005

① Wikipedia, Normal Distribution, Properties

X, Y ind. normal RV's w/ mean = 0 variance σ_x^2, σ_y^2

The $P(z)$ where $z = XY$ is

$$p(z) = \frac{1}{\pi \sigma_x \sigma_y} K_0\left(\frac{|z|}{\sigma_x \sigma_y}\right)$$

K_0 is a modified Bessel function

② GR p 684 6.561.16

$$\int_0^\infty x^\mu K_\nu(ax)\,dx = 2^{\mu-1} a^{-\mu-1}\, \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right)$$

variance: $\mu = 2$, $\nu = 0$ $\quad$ $\mathrm{Re}(\mu+1\pm\nu) > 0$ $\quad$ $\mathrm{Re}(a) > 0$

$$2\int_0^\infty x^2 K_0(ax)\,dx = 2 \cdot 2^1\, a^{-3}\, \Gamma\left(\tfrac{3}{2}\right) \Gamma\left(\tfrac{3}{2}\right)$$

$$= 4\, a^{-3} \left\{\Gamma\left(\tfrac{3}{2}\right)\right\}^2$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

so $\quad 2\int x^2 K_0(ax)\,dx = 4\, a^{-3} \frac{\pi}{4} = \pi a^{-3}$

and $\quad$ var $p(z)$ is $\quad \frac{1}{\pi \sigma_x \sigma_y} \cdot \pi \sigma_x^3 \sigma_y^3 = \sigma_x^2 \sigma_y^2$

③ consider sum of N random variables. given additivity rule for independent random variables

$$\mathrm{var}\left(\sum_{i=1}^{N} x_i y_i\right) = N \sigma_x^2 \sigma_y^2$$