Statistics of
detects using cross-correlation
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signal $A_i$ of length $N_i$ it has mean-squared amplitude $a^2$ with $\sum_i A_i^2 = N_a^2$ ($A_i$ has zero mean)
signal $B_i$ also of length $N_i$ with mean-squared amplitude $b^2$ with $\sum_i B_i^2 = N_b^2$
$B_i$ has zero mean
noise $N_i$ of length $N_i$, uncorrelated, normally distributed, zero mean, variance $n^2$

consider cross-correlation

$$\sum_i A_i (B_i + N_i) = \sum_i A_i B_i + \sum_i A_i N_i$$

now assume that $A_i \propto B_i$ (same signals, different ampl) and assume $A_i$ is uncorrelated w.r.t. $N_i$, so it can be treated as r.v. in second term. Then first term has constant value $N_a b$ and second term has variance $N_a n^2$.

Let's say that a detection occurs when

Term1 > $2 \sqrt{\text{Variance of term 2}}$ i.e. 95% conf

or

$N_a b > 2 \sqrt{N} a n$

or

$\frac{1}{2} \sqrt{N} \frac{b}{n} > 1$
Effect of Bandwidth

Note correlation between \((X; \tilde{N}_i)\) and \((X, \tilde{N}_i, \tilde{N}_{i+1})\) will make term2 longer than the uncorrelated case since \(\text{Var}(Z, \tilde{Z}_i) = \text{Var}(Z) + \text{Var}(\tilde{Z}_i) + 2\text{Cov}(Z, \tilde{Z}_i)\).

Special (extreme) case of complete correlation. Note \(\text{Var}(Z) = c^2 \text{Var}(\tilde{Z})\) so if \(Z_i = A_i \tilde{Z}_i\) and all's \(\tilde{Z}^i\)'s are fully correlated.

\[\text{Var}(Z, \tilde{Z}_i) = \text{Var}(N \tilde{Z}_i) = N^2 \text{Var}(\tilde{Z}_i) = N^2 c^2 N^2\]

In this case, the discriminant's condition is

\[Nab > 2 Nan\]

or

\[\frac{1}{2} N \frac{b}{n} > 1\]

That is, there is no improvement in discrimination with number of points.

In the more general case of bandwidth, \(w_i\), where \(w_0\) means fully correlated and \(w_1\) means fully uncorrelated, we might expect a formula something like

\[\frac{1}{2} N \frac{w}{n} \frac{b}{n} > 1\]
which at least have the right limits. The exact formula will depend upon the exact definition of "bondwidth". More exactly, the autocorrelation function of $z_i$.

This follows from

$$\text{Var}(\sum z_i) = [1, 1, \ldots] \text{cov}(z_i) \begin{bmatrix} \vdots \end{bmatrix} = \sum_i \sum_j \text{cov}(z_i)_{ij}$$

suppose $[\text{cov}(z_i)]_{ij} \propto \exp\left\{-d|i-j|\right\}$

Then $\text{Var}(\sum z_i) = \sum_i \sum_j \exp\left\{-d|i-j|\right\}$

This can be divided into the diagonal part of the sum (that is $i = j$), which is just $N$ plus $2$ times the upper triangle part (where $j > i$).

Note along the way, this is $\frac{N}{i=1} \exp(-jc)$

$$= \sum_{j=1}^{N} (e^{-c})^j = \sum_{j=1}^{N} E^j \text{ with } E = e^{-c}$$

Now note $6 \epsilon R p 1 < 0.118.0$

$$\sum_{k=1}^{n} a q^{k-1} = a \left( \frac{q^n - 1}{q - 1} \right)$$
with \(a=1\) and \(q=e^{-c}\) this gives
\[
\sum_{k=1}^{N} (e^{-c})^{k-1} = \frac{e^{-Nc} - 1}{e^{-c} - 1}
\]
but this includes the zero order term \((e^{-c})^0 = 1\). Subtract it
\[
\sum_{k=2}^{N} (e^{-c})^{k-1} = \frac{1-e^{-Nc}}{1-e^{-c}} - \frac{1-e^{-c}}{1-e^{-c}} = \frac{e^{-c} - e^{-Nc}}{1-e^{-c}} = f(c, N)
\]
so the \(\Xi_j\) would be something like
\[
N + 2f(c, N)
\]
ignoring the fact that all the rows are not quite the same as the first. So, the detection condition would be
\[
N_{ab} > 2 \sqrt{N + 2f(c, N)}
\]

\[
\frac{1}{2} \frac{N}{\sqrt{N + 2f}} \cdot \frac{1}{n} > 1
\]
but this is an approximation.
exact soln

\[ \text{main diagonal } \sum = N \]

\[ \text{first diagonal } \sum = (N-1) e^{-c} \]

\[ \text{2nd diag } \sum = (N-2) e^{-2c} \]

\[ \text{last diag } \sum = 0 \cdot 1 e^{-(N-1)c} \]

\[ \text{total sum has for } \sum (N-j+1)(e^{-c})^{j-1} \]

\[ \text{Term 1} = (N+1) \sum \frac{(e^{-c})^{j-1}}{j=1} \]

\[ \text{Term 2} = \sum \frac{N}{j=1} j(e^{-c})^{j-1} \]

Term 1 is just from before \((N+1) \frac{1-e^{-nc}}{1-e^{-c}}\)

For term 2 see G+R 0.113 with \(a=0\) and \(r=1\)

\[ \sum_{k=0}^{N-1} k q^k = -\frac{(N-1) q^N}{1-q} + \frac{q (1-q^{N-1})}{(1-q)^2} \]

The fact that the sum starts at 0 rather than 1 is unclear since \(k=0\) appears in leading term so

\[ \sum_{k=1}^{N} k q^k = -\frac{N q^N}{1-q} + \frac{q (1-q^{N-1})}{(1-q)^2} = 0 \]

\[ \frac{q (1-q^N)}{(1-q)^2} - \frac{N q^N (1-q)}{(1-q)^2} = \frac{q - q^{N+1} - N q^N + N q^{N+1}}{(1-q)^2} \]
\[ = \frac{(N-1)q^{N+1} - Nq^N + q}{(1-q)^2} = g_{c(c,N)} \quad \text{with} \quad q = e^{-c} \]

So \( \sum_{i=1}^{N} C_{ij} \) is twice this minus \( N \) (since we've counted the main diagonal twice) \((2g_{c(c,N)} - N)\)

and detection criteria is

\[ \frac{1}{2} \frac{N}{\sqrt{2g - N}^2} > 1 \]