2/15 – 2/22/98

MRN 073

Shear wave splitting

Let $\underline{n}$ and $\underline{e}$ be north and east seismograms with length, n, and sampling, δt.

Let $\underline{x}$ and $\underline{y}$ be two new seismograms, constructed from $\underline{n}$ and $\underline{e}$ by a rotation, θ:

$$\underline{x}(\theta) = \underline{n}\cos\theta - \underline{e}\sin\theta$$
$$\underline{y}(\theta) = -\underline{n}\sin\theta + \underline{e}\cos\theta$$

Let S_x be the r.m.s. amplitude of $\underline{x}$: $S_x^2 = \underline{x}^T\underline{x}/n$. Similarly for S_y.

Let $\underline{x}_\tau$ be $\underline{x}$ delayed by time, τ.

The shear wave splitting assumption is that there is a θ, τ such that $\underline{x}_\tau(\theta)/S_x = \underline{y}(\theta)/S_y$.

Suppose $\underline{x}$ and $\underline{y}$ have additive noise $\underline{n}_x$, $\underline{n}_y$, assumed to be uncorrelated with variance σ_x^2, σ_y^2.

Then the error

$$\underline{e} = \frac{1}{\sqrt{2}}\left(x_\tau(\theta)/S_x - y(\theta)/S_y\right)$$ has variance $\sigma_e^2 = \frac{1}{2}\left(\sigma_x^2/S_x^2 + \sigma_y^2/S_y^2\right)$.

defining a signal-to-noise ratio $r_x = s_x / \sigma_x$ and similarly for r_y, then assuming $r_x \approx r_y \approx r$, $\sigma_c^2 = r^{-2}$.

Variance of a linear inverse problem

Let $\underline{\underline{G}}\,\underline{m} = \underline{d}$ be the linear inverse problem, with least squares solution
$$\underline{m}^{est} = \underline{\underline{G}}^{-g}\underline{d}, \quad \underline{\underline{G}}^{-g} = [\underline{\underline{G}}^T\underline{\underline{G}}]^{-1}\underline{\underline{G}}^T$$

If $\underline{d}$ contains uncorrelated noise with variance σ_d^2, then
$$\underline{\underline{\text{cov}\,m}} = \sigma_d^2\, \underline{\underline{G}}^{-g}\underline{\underline{G}}^{-gT} = \sigma_d^2\, [G^TG]^{-1}$$

Let the rms data amplitude be $s_d = \underline{d}^T\underline{d} / n$. Then the error $\underline{e} = (\underline{d} - \underline{G}\,\underline{m}) / s_d$ has variance
$$\sigma_e^2 = \sigma_d^2 / s_d^2 = r_d^{-2}$$

Let the total error $E = \underline{e}^T\underline{e} / n$, then note $\nabla_m\nabla_m E = 2[G^TG] / (n s_d^2)$ (by simple differentiation). Thus
$$[G^TG]^{-1} = \frac{1}{n s_d^2}\left[\tfrac{1}{2}\nabla_m\nabla_m E\right]^{-1}_{\underline{m}=\underline{m}^{est}}$$

and

$$\underline{\underline{\text{cov}\, m}} = \frac{\sigma_d^2}{n s_d^2} \left[\tfrac{1}{2} \nabla_m \nabla_m E \right]^{-1}_{\underline{m}=\underline{m}^{est}}$$

$$= \frac{1}{n r_d^2} \left[\tfrac{1}{2} \nabla_m \nabla_m E \right]^{-1}_{\underline{m}=\underline{m}^{est}}$$

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now let us apply this linear approximation to the shear wave splitting problem. we note

$$E = \underline{e}^T \underline{e} / n = \frac{1}{2} \left(\frac{\underline{x}_\tau^T \underline{x}_\tau}{n s_x^2} + \frac{\underline{y}^T \underline{y}}{n s_y^2} - 2 \frac{\underline{x}_\tau^T \underline{y}}{n s_x s_y} \right)$$

$$= (1 - C)$$

where $C = \underline{x}_\tau^T \underline{y} / n s_x s_y$

Thus $\frac{1}{2} \nabla_m \nabla_m E = -\frac{1}{2} \nabla_{\underline{m}} \nabla_{\underline{m}} C$

and $\underline{\underline{\text{cov}\, m}} = -\frac{1}{n r^2} \left[\tfrac{1}{2} \nabla_{\underline{m}} \nabla_{\underline{m}} C \right]^{-1}_{m=m^{est}}$

but we have assumed that the errors are uncorrelated. If the seismograms are over-sampled by a factor f, then we should replace n with n/f:

$$\underline{\underline{\text{cov}\, m}} = \frac{-f}{n r^2} \left[\tfrac{1}{2} \nabla_{\underline{m}} \nabla_{\underline{m}} C \right]^{-1}_{\underline{m}=\underline{m}^{est}}$$

estimating r from c

$$c = \frac{(\underline{x} + n_x)^T (\underline{y} + \underline{n}_y)}{\|\underline{x} + \underline{n}_x\|_2^{1/2} \|\underline{y} + \underline{n}_y\|_2^{1/2}}$$

has expected value $\langle c \rangle$

$$\langle c \rangle \approx \frac{\langle \underline{x}^T \underline{y} \rangle}{[\langle \underline{x}^T \underline{x} \rangle + \langle \underline{n}_x^T \underline{n}_x \rangle]^{1/2} [\langle \underline{y}^T \underline{y} \rangle + \langle n_y^T n_y \rangle]^{1/2}}$$

$$= \frac{\langle \underline{x}^T \underline{y} \rangle}{(\underline{x}^T \underline{x})^{1/2} (y^T \underline{y})^{1/2} \left[1 + \frac{\langle n_x^T n_x \rangle}{\underline{x}^T \underline{x}}\right]^{1/2} \left[1 + \frac{\langle n_y^T n_y \rangle}{\underline{y}^T \underline{y}}\right]^{1/2}}$$

$$\approx \frac{c^{\text{true}}}{1 + r^{-2}} = \frac{1}{1 + r^{-2}}$$

so given observed c, $r = (c^{-1} - 1)^{-1/2}$

eg if $c = 0.9$ then $r = 3$

estimating f from $\underline{e}$

compute autocorrelation of $\underline{e}$ and assume $f \approx$ halfwidth of main peak.

could also estimate $\bar{f}^2 = e^T e / n$

$$c = A + B\phi + C\phi^2 + D\tau + E\tau^2 + F\tau\phi$$

$$\frac{\partial c}{\partial \phi} = 0 = B + 2C\phi + F\tau$$

$$\frac{\partial c}{\partial \tau} = D + 2E\tau + F\phi$$

$$\begin{pmatrix} F & 2C \\ 2E & F \end{pmatrix} \begin{pmatrix} \tau \\ \phi \end{pmatrix} = \begin{pmatrix} -B \\ -D \end{pmatrix}$$

$$\frac{1}{F^2 - 4EC} \begin{pmatrix} F & -2C \\ -2E & F \end{pmatrix} \begin{pmatrix} -B \\ -D \end{pmatrix} = \begin{pmatrix} \tau \\ \phi \end{pmatrix} =$$

$$[2CD - BF, \; -2BE - DF]^T / \det$$