Inverse rule

Propagator w/ source term

Menke 04-24-11

MRN076

$$m(z) = P(z, z_0)\, m(z_0) = P(z, z_0)\, m_0$$

$$m_0 = P(z_0, z)\, m(z) = P^{-1}(z, z_0)\, m(z)$$

$$m(z) = P(z, z_0) P(z_0, z)\, m(z) \quad \text{so} \quad P^{-1}(z_0, z) = P(z, z_0)$$

=

$$P(z, z_0) = P(z, z_1) P(z_1, z_0)$$

Product rule

=

$$\partial_z P^{-1}(z, z_0) = -P^{-1}(z, z_0) A$$

Derivative rule

start with wave eqn

$$\partial_z m(z) - A\, m(z) = 0$$

$\times P^{-1}(z, z_0)$

$$P^{-1}(z, z_0) \frac{\partial}{\partial z} m(z) - P^{-1}(z, z_0) A\, m(z) = 0$$

$$\frac{\partial}{\partial z}\left(P^{-1}(z, z_0)\, m(z) \right) = \frac{\partial}{\partial z}\left(P(z_0, z)\, m(z) \right) = \frac{\partial}{\partial z}(m_0) = 0$$

chain rule

$$= \left[\frac{\partial}{\partial z} P^{-1}(z, z_0) \right] m(z) + P^{-1}(z, z_0) \frac{d}{dz} m(z)$$

$$\text{so } P^{-1}(z, z_0) \frac{d}{dz} m(z) = -\left[\frac{\partial}{\partial z} P^{-1}(z, z_0) \right] m(z)$$

$$\text{and } -\left[\frac{\partial}{\partial z} P^{-1}(z, z_0) \right] m(z) - P^{-1}(z, z_0) A\, m(z) = 0$$

= inhomogenous solution

$$\partial_z m(z) - A\, m(z) = f$$

$\times P^{-1}(z, z_0)$

$$P^{-1}(z, z_0) \frac{\partial}{\partial z} m(z) - \underbrace{P^{-1}(z, z_0) A\, m(z)}_{\text{product rule}} = P^{-1}(z, z_0) \underline{f}$$

$$\underbrace{\qquad + \left[\partial_z P^{-1}(z, z_0) \right] m}_{\text{chain rule}}$$

$$\frac{d}{dz}\left[P^{-1}(z, z_0)\, m(z) \right] = P^{-1}(z, z_0)\, \underline{f}$$

integrate, with integration constant, c.

$$P^{-1}(z, z_0)\, m(z) = C + \int_{z_0}^{z} P^{-1}(z', z_0)\, f(z')\, dz'$$

when $z=0$

$$P^{-1}(z_0, z_0)\, m_0 = 0 + c$$

so $C = m_0$

$\times\ P(z, z_0)$ with $P^{-1}(z', z_0) = P(z_0, z')$

$$m(z) = P(z, z_0)\, m_0 + \int_{z_0}^{z} P(z, z_0) P(z_0, z')\, f(z')\, dz'$$

$$= P(z, z_0)\, m_0 + \underbrace{\int_{z_0}^{z} P(z, z')\, f(z')\, dz'}_{I}$$

Prop. as source at z' to z, for all z''s

e.g. $f(z') = \delta(z - z_s) F$

$I(z) = P(z, z_s) F$

= jump discontinuity

$$f(z) = F_1\, \delta(z - z_s) + F_2\, \partial_z\, \delta(z - z_s)$$

$$I = \int_{z_0}^{z} P(z, z')\, \delta(z' - z_s)\, dz'\, F_1 + \int_{z_0}^{z} P(z, z') \frac{\partial}{\partial z}\, \delta(z' - z_s)\, dz'\, F_2$$

use derivative rule

$I(z_s^-) = 0$

$$I(z_s^+) = P(z_s, z_s) F_1 - \left[\frac{\partial}{\partial z'} P(z_s, z')\right]_{z' = z_s} F_2$$

$$= F_1 + F_2\, P(z_s, z_s)\, A(z_s)$$

$$= F_1 + F_2\, A(z_s)$$