Adjoint Method of Calculating Data Kernel.

given

1. a field $u(t)$ satisfies the differential equation $A u(t) = f(t)$ with known boundary conditions. A is a linear differential operator

2. a datum, d_i, is a linear functional of the field $u(t)$
$d_i = (l_i(t), u(t)) \equiv \int l_i(t)\, u(t)\, dt$

3. The data kernel $g_i(t)$ relates a perturbation δf to a perturbation δd_i : $\delta d_i = (g_i(t), \delta f)$

Then

4. $d_i = (l_i, u) = (l_i, A^{-1} f) = (A^{-1*} l_i, f) = (A^{*-1} l_i, f)$
where A^* is the adjoint to A and we have used the identity $A^{*-1} = A^{-1*}$

5. $\delta d_i = (A^{*-1} l_i, \delta f)$ so the data kernel is $g_i = A^{*-1} l_i$
and solves $A^* g_i = l_i$

6. Written as a matrix eqn, a causal problem must be lower triangular:

$$\begin{pmatrix} \times & & & & \\ \times & \times & & 0 & \\ \times & \times & \times & & \\ \times & \times & \times & \times & \\ \cdots & & & & \end{pmatrix} \begin{pmatrix} u \end{pmatrix} = \begin{pmatrix} f \end{pmatrix} = A u$$

backsolving is equivalent to solution via "shooting forward in time".

7. Similarly, the adjoint equation, which involves the transpose of the above matrix, is upper triangular

$$\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \\ 0 & & & \vdots \end{pmatrix} \begin{pmatrix} g_i \end{pmatrix} = \begin{pmatrix} l_i \end{pmatrix} = A^* g_i$$

backsolving is equivalent to solving $A^* g_i = l_i$ by shooting backward in time.