Covariance of Fourier Coefficients

$x_i\ ,\ i=1,N$ gaussian dist w/ $\bar{x}=0$ $[\mathrm{cov}\,x]=\sigma_x^2 I$

CASE A: $$x_k = \sum_{j=1}^{\frac{N}{2}+1} A_n \cos(\omega_j t_k) + B_n \sin(\omega_j t_k)$$

let $\underline{a} = [A_1, A_2, B_2 \cdots]^T$

if $\underline{x} = \underline{\underline{F}}\,\underline{a}$ Then $(\underline{\underline{F}}^T \underline{\underline{F}})^{-1} = \frac{2}{N}\underline{\underline{I}}$ so $\mathrm{cov}(a) = \frac{2}{N}\sigma_x^2 \underline{\underline{I}}$

CASE B: $$x_k = \frac{1}{N}\sum_{-\omega_{NY}}^{+\omega_{NY}} c_j \cdot e^{i\omega_j t_k}$$

note $c_r = \frac{N}{2} A$ $c_i = \frac{N}{2} B$

so cov (real or imag parts of c) $= \frac{N^2}{4}\frac{2}{N}\sigma_x^2 I = \frac{N}{2}\sigma_x^2 \underline{\underline{I}}$

spectrum

CASE A $\quad s = A^2 + B^2$

mean $\chi_\nu^2 = \nu$ $\quad$ var $\chi_\nu^2 = 2\nu$

for x's of variance σ^2, mult. by σ^4

so $(\mathrm{var}\, s) = 4\sigma_a^4 = 4\cdot\left(\frac{4}{N^2}\sigma_x^4\right) = \frac{16}{N^2}\sigma_x^4$

and $\sigma_s = \frac{4}{N}\sigma_x$

CASE B $\quad s = c_r^2 + c_i^2$

$(\mathrm{var}\, s) = 4\sigma_c^4 = 4\cdot\left(\frac{N}{2}\sigma_x^2\right)^2 = N^2\sigma_x^4$

$\sigma_s = N\sigma_x^2$

Variance of a spectral peak

CASE A

$$S = A^2 + B^2 = (A_0 + \delta A)^2 + (B_0 + \delta B)^2$$

$$= A_0^2 + B_0^2 + 2A\delta B + 2B\delta A$$

$$\Delta S = 2A\delta B + 2B\delta A$$

$$\mathrm{Var}\,\Delta S = 4A^2\,\mathrm{Var}\,\delta B + 4B^2\,\mathrm{Var}\,\delta A$$

$$= 4 S_0 \,\mathrm{Var}(a)$$

$$= 4 S_0 \frac{2}{N} \sigma_x^2 = \frac{8 S_0}{N} \sigma_x^2$$

$$\sigma_{\Delta S} = \frac{2\sqrt{2}\sqrt{S_0}\,\sigma_x}{\sqrt{N}}$$

CASE B

$$S = C_r^2 + C_i^2$$

$$\mathrm{Var}\,\Delta S = 4 S_0 \,\mathrm{Var}(c)$$

$$= 4 S_0 \frac{N}{2} \sigma_x^2$$

$$= 2 N S_0 \sigma_x^2$$

$$\sigma_{\Delta S} = \sqrt{2}\sqrt{N}\sqrt{S_0}\,\sigma_x$$