Investigations of the equation

$$t_{ij} = t_i^0 + \sum_k d_{ijk} S_k$$

Note: I introduce the symbol $w_i = 1$ for all $i$ in order to "match indices" in the equation.

**Eqn. 2**

$$t_{ji} = t_j^0 w_j + \sum_k d_{jik} S_k$$

Now eliminate $t_j^0$

$$\sum_i \sum_j t_{ij} = \sum_i N t_i^0 + \sum_k \sum_j d_{ijk} S_k$$

With $N = \sum w_j$

Solve:

$$t_i^0 = \frac{1}{N} \sum_j t_{ij} = -\frac{1}{N} \sum_k \sum_j d_{jik} S_k$$

**Eqn. 3**

Sub:

$$t_{ij} - \frac{w_j}{N} \sum_k t_{ik} = \sum_k \left( d_{ijk} - \frac{w_j}{N} \sum_p d_{ipk} \right) S_k$$

$$= \sum_k \left( \delta_{jp} d_{ipk} - \frac{w_j}{N} \sum_p d_{ipk} \right) S_k$$

$$= \sum_p \left( \delta_{jp} - \frac{w_j w_p}{N} \right) \sum_k d_{ipk} S_k$$

Note: Addition of $c_{kp} = 1$ is just to match indices.

And $\delta_{kp}$ = Kronecker delta

$$= \frac{1}{N} \sum_p \left( NS_{jp} - w_j w_p \right) \left( \sum_k d_{ipk} S_k \right)$$

**Eqn. 4**

Note: matrices $B_{ij}$ and $C_{pq}$ are in the form

$$A_{ii} = \frac{1}{N} \sum_i B_{ip} C_{pj}$$

$$B_{ip} = \sum_k d_{ipk} S_k \quad \text{and} \quad C_{pq} = C_{qp} = NS_{jp} - w_j w_p$$
Note that \( C \) is the symmetric matrix

\[
C = \begin{pmatrix}
(n-1) & -1 & -1 & \cdots \\
-1 & (n-1) & -1 & \cdots \\
-1 & -1 & (n-1) & -1 & \cdots
\end{pmatrix}
\]

Now let's consider when \( BC = 0 \). Clearly, if \( B \) has constant rows, then this is true; \( B = b_i \mathbf{w}_p \) where \( b_i \) is an arbitrary vector.

\[
0 \overset{?}{=} \frac{1}{P} \sum_{p} B_{ip} C_{pj} = \frac{1}{P} \sum_{p} b_i \mathbf{w}_p (N \delta_{jp} - w_j \mathbf{w}_p) \\
= \frac{1}{P} \sum_{p} b_i \mathbf{w}_p N \delta_{jp} - \frac{1}{P} \sum_{p} b_i \mathbf{w}_j \mathbf{w}_p \overset{\text{note } \frac{1}{P} \mathbf{w}_p^2 = N}{\Rightarrow} \\
= N b_i \mathbf{w}_j - N b_i \mathbf{w}_j = 0
\]

Now let's construct the \( S_k \) that solves

\[
B_{ip} = b_i \mathbf{w}_p = \frac{1}{P} \sum_{k} d_{ip} d_{kp} S_k
\]

Let's define \( d^{-1} \) to be the inverse of \( d \), in the sense that \( \frac{1}{P} \sum_{p} d_{ip} d_{kp} = \delta_{ik} \). Then

\[
\frac{1}{P} \sum_{p} d_{ip} \mathbf{w}_p = \frac{1}{P} \sum_{k} \frac{1}{P} \sum_{p} d_{ip} d_{kp} S_k
\]

\[
= \frac{1}{P} \sum_{k} \delta_{ik} S_k = S_k
\]

So any solution of the form \( S_k = \frac{1}{P} \sum_{p} d_{ip} \mathbf{w}_p \) when \( b_i \) is an arbitrary vector is a null solution of the equation.
If a solution $s_k$ is perturbed by adding a null solution

$$s_k = s_k^0 + s_k^{null} = s_{new}$$

and it the corresponding $t_i^k$ is recalculated using equation 3,

$$t_i^{new} = f(s_{new})$$

Then these new solutions also satisfy equation 1.

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Note that the existence of $d^{-1}$ is implied by the equation

$$t_i^j = \sum_k d_{ij}^k s_k$$

(i.e. Eqn 1 without the $t_0$ term) being solvable. In the tomography case, that's the tomography w/o the source statics case. That seems like a fairly "weak" = reasonable condition.