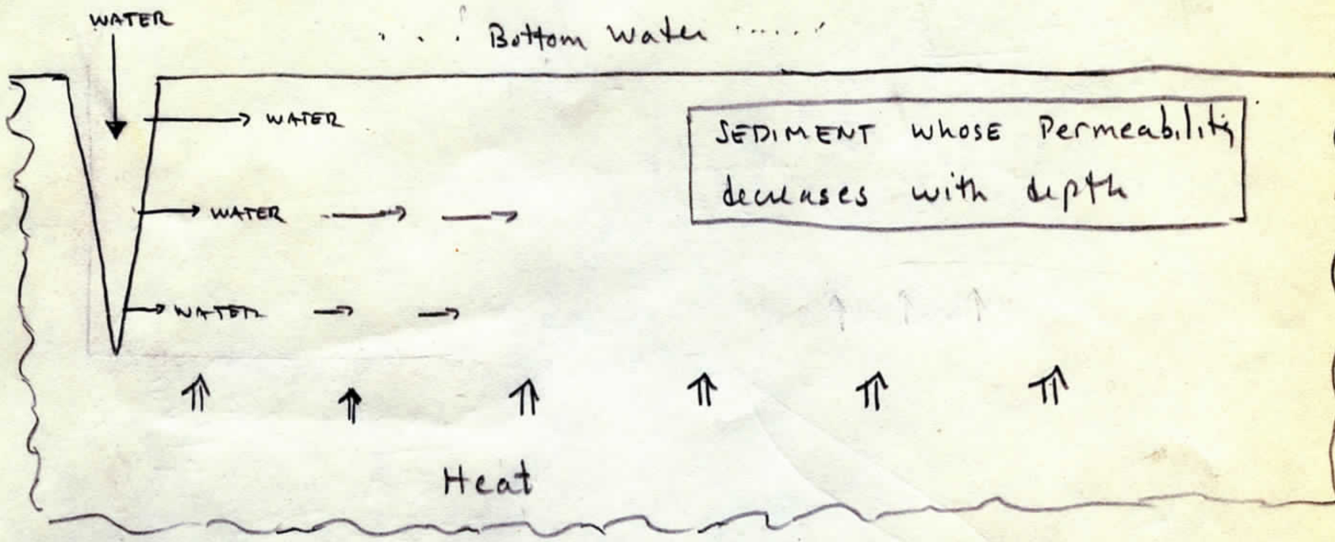
The affect of horizontal water movement in sediment on the sediment's temperature profile.

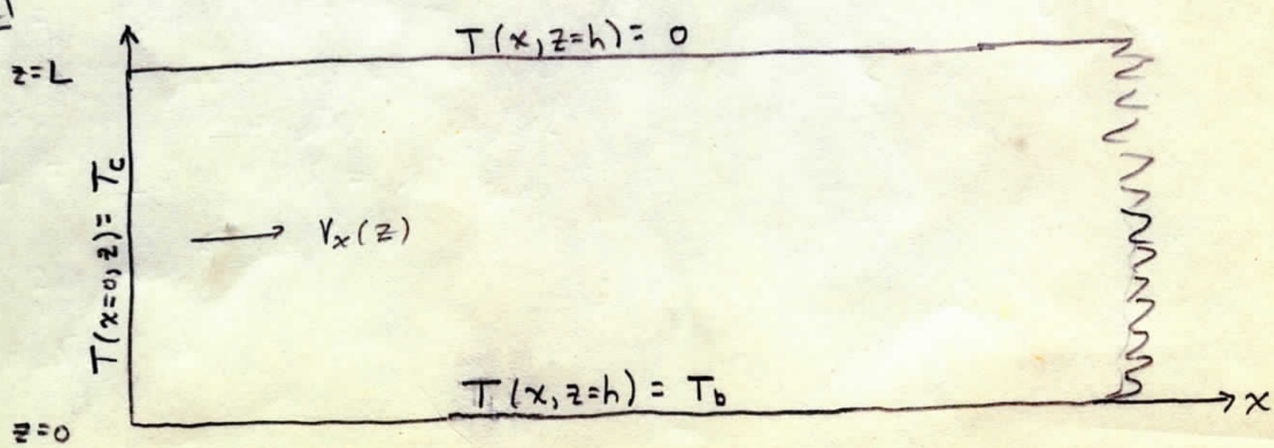
The situation



Assumptions and approximations

0. situation doesn't change with time.
1. WATER velocity horizontal, variation with depth known
2. horizontal flow of heat due to water movement only
3. vertical flow of heat due to conduction only
4. Thermal properties of sediment uniform

model



note $T_0 \geqslant T_c \geqslant 0$

Heat flow equation $\rho c \left(\frac{\partial T}{\partial t} + \underline{V} \cdot \nabla T \right) = k \nabla^2 T$

simplified by assumptions $\rho c \, V_x(z) \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial z^2}$

general solution of form: $T(x, z) = A(x) B(z) + C_1 z + C_0$

plug into equation and rearrange to get :

$$\frac{k}{\rho c} V_x^{-1}(z) \frac{1}{B(z)} \frac{d^2 B(z)}{dz^2} = -m^2 = \frac{1}{A(x)} \frac{d A(x)}{dx}$$

($-m^2$ The separation constant) this gives two equations for $A(x)$ and $B(z)$:

$$\frac{d A(x)}{dx} = -m^2 A(x)$$

$$\frac{d^2 B(z)}{dz^2} = -m^2 \frac{\rho c}{k} V_x(z) B(z)$$

the equation for $A(x)$ can be immediately solved to obtain $A(x) = e^{-m^2 x}$

The equation for $B(x)$ cannot be solved until The function $V_x(z)$ is specified.

Special Case : it is instructive to solve the simple case of $V_x(z) = \text{constant} = v$. Then if $\epsilon = \frac{\rho c v}{k}$

$$B(z) = C_2 \sin(m\epsilon^{\frac{1}{2}} z) + C_3 \cos(m\epsilon^{\frac{1}{2}} z)$$

and the general solution is

$$T(x,z) = C_0 + C_1 z + [C_2 \sin(m\epsilon^{1/2} z) + C_3 \cos(m\epsilon^{1/2} z)]\, e^{-m^2 x}$$

The constants are chosen by applying the boundary conditions. To have the temperature on the top and bottom surfaces to be 0 and T_b respectively we could set

$$C_0 + C_1 z = T_b - \frac{T_b}{L} z \; ,$$

we must then make sure that the sine and cosine make no contribution to the temperature at the top and bottom surfaces. This is done by setting $C_3 = 0$ and $m = \frac{n\pi}{\epsilon L}$. Then $\sin(z=0) = 0$ and $\sin(z=L) = \sin\left(\frac{n\pi L}{L}\right) = 0$ if n is an integer. The solution is then

$$T(x,z) = T_b - \frac{T_b}{L} z + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi z}{L}\right) e^{-\frac{n^2\pi^2}{\epsilon L^2} x}$$

The last boundary condition to satisfy is the one on the left hand edge (the crack) ; $T(x=0,z) = T_c$ this condition implies :

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi z}{L}\right) = -(T_b - T_c) + \frac{T_b}{L} z$$

The coefficients can be found using fourier's Theorem, namely

if $\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi z}{L}\right) = f(z)$ then $C_n = \frac{2}{L}\int_0^L f(z)\sin\left(\frac{n\pi z}{L}\right) dz$

First the term $-(T_b - T_c)$

$$C_n^1 = -\frac{2}{L}(T_b - T_c)\int_0^L \sin\left(\frac{n\pi z}{L}\right) dz = +\frac{2}{L}(T_b - T_c)\frac{L}{n\pi}(\cos n\pi - 1)$$

$$= \begin{cases} \dfrac{-4(T_b - T_c)}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$C_n^2 = \frac{T_b}{L}\frac{2}{L}\int_0^L z \sin\left(\frac{n\pi z}{L}\right) dz = \frac{T_b}{L}\left[\frac{L^2}{n^2\pi^2}\sin\frac{n\pi z}{L} - \frac{zL}{n\pi}\cos\frac{n\pi z}{L}\right]_0^L$$

$$= \left(\frac{2T_b}{L^2}\right)\left(-\frac{L^2}{n\pi}\right)(\cos n\pi) = \frac{2T_b}{n\pi}(-1)^{n+1}$$

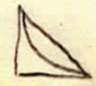
$X_{characteristic} = \frac{\epsilon L^2}{\pi^2}$

so final solution is

$\frac{4}{\pi} = 1.273$ $\quad -\frac{2}{\pi} = -0.636$

$$T(x,z) = T_b - \frac{T_b}{L}z + \frac{1}{\pi}\left(4T_c - 2T_b\right)\sin\frac{\pi z}{L}\, e^{-\frac{\pi^2}{\epsilon L^2}x}$$

$$+ \left[\sum_{n=3,5,7,9}^{\infty} \frac{-4(T_c - T_0)}{n\pi}\sin\frac{n\pi z}{L} + \sum_{n=2,3,4}^{\infty}(-1)^{n+1}\frac{2T_b}{n\pi}\sin\frac{n\pi z}{L}\right] e^{-\frac{\pi^2 n^2}{\epsilon L^2}x}$$

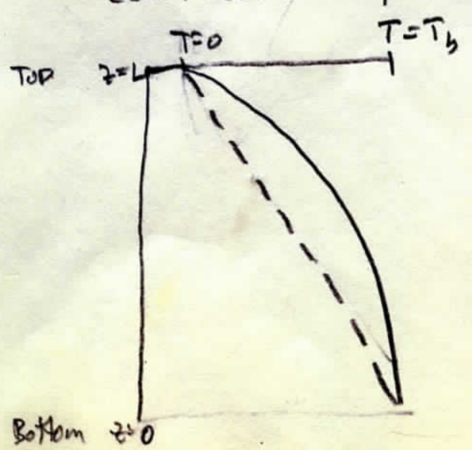
at X_{char} $T(z=L/2) = 0.5$ if $v=0$ if $v \neq 0$ $T(z=L/2) = 0.27$  quite significant examination of solution. note.

1. at $T(x, z=L)$ all sines are 0 and $T = T_b - \frac{T_b}{L}L = 0$
2. at $T(x, z=0)$ all sines are 0 and $T = T_b$
3. at $T(x=0, z)$ all exponentials are 1 and fourier series add up to give $T = T_c$
4. as $x \to \infty$ exponentials decay towards zero and Temperature is a function only of , and is linear. $T = T_b - \frac{T_b}{L} z$. This is reasonable since at large x the water equilibrates to the temperature of the sediment and no longer affects the gradient.
5. at large but finite values of x the temperature profile is modified by the $n=1$ term only, since higher n exponentials decay faster. There are 2 cases:

case 1

$T_c > \frac{1}{2} T_b$

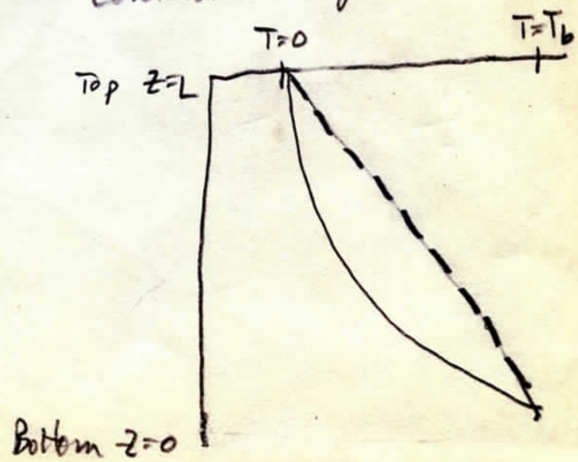
contribution of sine positive



case 2

$T_c < \frac{1}{2} T_b$

contribution of sine negative



A more general case of the velocity function $V_x(z)$.

suppose V increases upward in the sediment rapidly, which is reasonable since the sediment's permeability increases upward. A reasonable choice of $V_x(z)$ is

$$V_x(z) = V\left(\frac{z}{L}\right)^p \quad ; \quad p \text{ a positive integer}$$

Then the equation for $B(z)$ is

$$\frac{d^2 B(z)}{dz^2} = -\frac{m^2 \epsilon}{L^p} z^p B(z)$$

$$\text{where} \quad \epsilon = \frac{n \rho c V}{k}$$

Then the solution is [Hildebrand p. 156]

$$B(z) = C_\pm z^{1/2} J_{\pm\frac{1}{p+2}}\left(\frac{2m\epsilon^{\frac{1}{2}}}{(p+2)L^{p/2}} z^{\frac{p+2}{2}}\right) = C_\pm z^{1/2} J_{\pm\frac{1}{p+2}}\left(\frac{2m\epsilon^{\frac{1}{2}}L}{(p+2)}\left(\frac{z}{L}\right)^{\frac{p+2}{2}}\right)$$

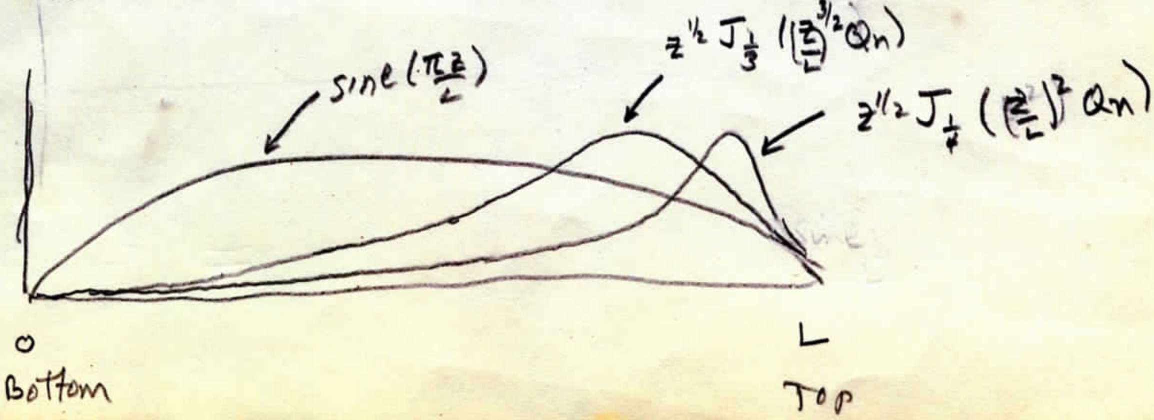
as in the case of $V_x(z) =$ constant, we can set one of the constants equal to zero so the boundary conditions are satisfied. in this case it is C_- since $J_{-\frac{1}{p+2}}\left(\frac{2m\epsilon}{(p+2)L^{p/2}} z^{\frac{p+2}{2}}\right)$ is singular at $z=0$. Then suppose Q_n is the n-th zero of the Bessel function $J_{\frac{1}{p+2}}$. Then we can make the Bessel function zero at the bottom boundary (its automatically zero at the top boundary) by setting

$$\frac{2m\epsilon^{\frac{1}{2}}L}{p+2} = Q_n \quad \text{or} \quad m = Q_n \frac{p+2}{2\epsilon^{\frac{1}{2}}L}$$

so the solution is:

$$T(x,z) = T_b - \frac{T_b}{L} z + \sum_{n=1}^{\infty} C_n z^{1/2} J_{\frac{1}{p+2}}\left(Q_n \left[\frac{z}{L}\right]^{\frac{p+2}{2}}\right) \exp\left(-\frac{Q_n^2 (p+2)^2}{4} \frac{x}{\epsilon L^2}\right)$$

To complete the formulae it would be necessary to expand the function $-(+T_b - T_c) + \frac{T_b}{L} z$ in the appropriate Bessel functions. However there are several observations we can make first. The $J_{\frac{1}{p+2}}$ will function much like the sine in the case of $V_x(z) = $ constant. The critical water temp. may not be $4T_c = 2T_b$ however, since this depends on the Fourier expansion. Never-the-less we expect some critical temperature in the range between 0 and T_b. The first Bessel function in the series will be added or subtracted from the linear gradient, depending on whether T_c is $<$ or $>$ critical. Again, at large x, the higher terms are exponentially small. The shape of the Bessel function is different than the sine, since the $z^{\frac{p+2}{2}}$ - term tends to concentrate most of the action near $x = L$:



Expanding $-(T_b - T_c) + \frac{T_b}{L} z$ in a Fourier Bessel Series

First we need an orthogonality relation for the solutions of the equation

$$\frac{d^2 B}{dz^2} + m^2 t^2 z^P B = 0 \quad , \text{ with homogeneous boundary conditions at } z=0 \text{ and } z=L.$$

we write this equation in the form $\frac{d}{dz}\left(p(z) \frac{dy}{dz} \right) + [q(z) + \lambda\, r(z)] y = 0$

by identifying [see Hildebrand p 206] :

$$p(z) = 1$$
$$q(z) = 0$$
$$\lambda = m^2 t^2$$
$$r(z) = z^P$$

Then the orthogonality rule is for two different solutions, say B_1 and B_2 $(\lambda_1 \neq \lambda_2)$ is :

$$\int_0^L B_1(z)\, B_2(z)\, z^P \, dz = 0$$

Expansion of a function $f(z)$ in terms of orthogonal solutions to the relevant equation

let $\quad B_n(z) = z^{1/2} J_{\frac{1}{P+2}} \left[Q_n \left(\frac{z}{L}\right)^{\frac{P+2}{2}} \right]$

then $\quad f(z) = \sum_{n=1}^{\infty} C_n\, B_n(z)$

and $\quad C_n = \dfrac{\int_0^L f(z)\, B_n(z)\, z^P\, dz}{\int_0^L B_n^2(z)\, z^P\, dz}$

or more specifically ;

$$C_n = \frac{\int_0^L f(z)\, J_{\frac{1}{p+2}}\left[Q_n \left(\frac{z}{L}\right)^{\frac{p+2}{2}}\right] z^{p+1/2}\, dz}{\int_0^L J^2_{\frac{1}{p+2}}\left[Q_n \left(\frac{z}{L}\right)^{\frac{p+2}{2}}\right] z^{p+1}\, dz}$$

The function $f(x)$ is $-(T_b - T_c) + T_b\left(\frac{z}{L}\right) = f(x)$.

suppose we let $f(x) = \alpha + \beta\left(\frac{z}{L}\right)$ and $\left(\frac{z}{L}\right) = \xi$ then

$z = \xi L$; when $z = 0$; $\xi = 0$; when $z = L$; $\xi = 1$ so

$$C_n = \frac{1}{L^{1/2}} \frac{\int_0^1 (\alpha + \beta\xi)\, J_{\frac{1}{p+2}}\left[Q_n \xi^{\frac{p+2}{2}}\right] \xi^{p+1/2}\, d\xi}{\int_0^1 J^2_{\frac{1}{p+2}}\left[Q_n \xi^{\frac{p+2}{2}}\right] \xi^{p+1}\, d\xi}$$

The presence of the $L^{-1/2}$ is to be expected, since it will complement the $z^{1/2}$ in the sum in the solution, so there will be only $\frac{z}{L}$'s in the solution. ie perfect scaling. The 3 integral that must be computed are

$$L_n = \int_0^1 J_{\frac{1}{p+2}}\left[Q_n \xi^{\frac{p+2}{2}}\right] \xi^{p+1/2}\, d\xi$$

$$M_n = \int_0^1 J_{\frac{1}{p+2}}\left[Q_n \xi^{\frac{p+2}{2}}\right] \xi^{p+3/2}\, d\xi$$

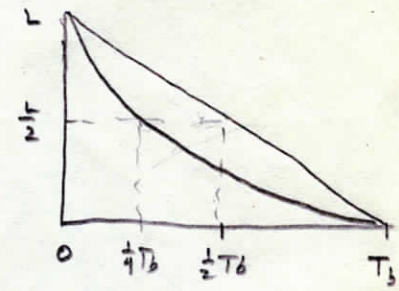
$$N_n = \int_0^1 J^2_{\frac{1}{p+2}}\left[Q_n \xi^{\frac{p+2}{2}}\right] \xi^{p+1}\, d\xi$$

L_n, M_n, and N_n are just numbers. The solution is

$$T(x,z) = T_b\left(1-\frac{z}{L}\right) \qquad A\,T_b + B\,T_c$$

$$+\sum_{n=1}^{\infty}\left[\frac{(M_n-L_n)T_b + L_n T_c}{N_n}\right]\left(\frac{z}{L}\right)^{1/2} J_{\frac{1}{p+2}}\left(Q_n\left(\frac{z}{L}\right)^{\frac{p+2}{2}}\right) e^{-\left(\frac{Q_n^2(p+2)^2}{4}\frac{x}{\epsilon L^2}\right)}$$

CHARACTERISTIC DISTANCES FOR v = constant core, $T_c = 0$



TO $\frac{1}{e}$ $\qquad x_c^{(1)} = \frac{\epsilon L^2}{\pi^2} = \frac{\rho c v L^2}{k\pi^2}$

amplitude of sine term $\quad -\frac{2T_b}{\pi e} = -0.23\,T_b$

TO $\frac{1}{e^2}$ $\qquad x_c^{(2)} = \frac{2\epsilon L^2}{\pi^2} = \frac{2\rho c v L^2}{k\pi^2}$

amplitude of sine term $\quad -\frac{2T_b}{\pi e^2} = -0.08\,T_b$

let $\quad \rho = 1.04$ gm/cm^3
c = heat capacity = 0.9 cal/gm °K
k = thermal cond = 1.6×10^{-3} cal/sec cm °K
v = hor. vel = 5×10^{-6} cm/sec
L = crack depth = 10^3 cm

$x_c^{(1)}$ = 2.6 meters

$x_c^{(2)}$ = 5.8 meters

	P=0	P=1	P=2	P=3
n=1	3.141 -1.414 +2.828	2.903 -0.908 +2.374	2.781 -0.670 +2.165	2.707 -0.531 +2.044
n=2	6.283 -1.000 0.000	6.033 -0.561 -0.458	5.906 -0.384 -0.645	5.830 -0.289 -0.744
n=3	9.425 -0.816 +1.632	9.170 -0.425 +1.251	9.042 -0.279 +1.112	8.965 -0.206 +1.042
n=4	12.566 -0.707 0.000	12.305 -0.349 -0.363	12.174 -0.224 -0.490	12.095 -0.162 +0.552
~~n=4~~	~~15.708~~	~~15.446~~	~~15.315~~	~~15.500~~

TABLE of Q_n

- - - -

C_{T_b}

- - - -

C_{T_c}

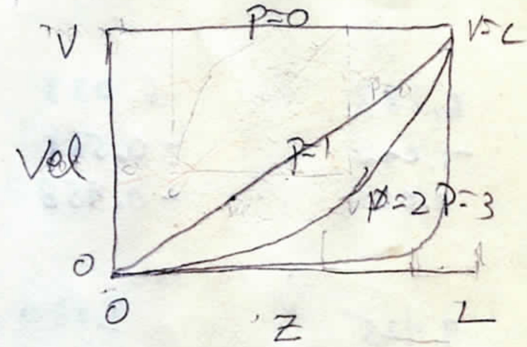
ref dep z=L

~~ref dep z=0~~

$AT_b + BT_c$

$T_c = \alpha T_B$ $\quad \alpha$ .1 .9

$V(z) = V\left(\frac{z}{L}\right)^p$



$(A + \alpha B)\, T_B$

	$\alpha = .1$	$\alpha = .9$
p=0	−1.13	+1.13
p=1	−0.67	+1.23
p=2	−0.45	+1.28
p=3	−0.33	+1.31