MRN133
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and addendum
to MRN020

$$\begin{array}{c} \leftarrow M \rightarrow \\ \begin{array}{c}\uparrow\\ N\\ \downarrow\end{array}\begin{bmatrix} \\ C \\ \\ \end{bmatrix}\end{array} = \begin{bmatrix} \\ a \\ \\ \end{bmatrix}_N [\leftarrow m \rightarrow]^M + \begin{bmatrix} \\ n \\ \\ \end{bmatrix}_N [- b -]^M + \begin{bmatrix} \\ D \\ \\ \end{bmatrix}$$

$$\underline{a} = \frac{1}{M}\underline{\underline{C}}\,\underline{m}^T - \frac{1}{M}(\underline{b}\,\underline{m}^T)\,\underline{n} \qquad (1)$$

$$\underline{b} = \frac{1}{N}\underline{n}^T\underline{\underline{C}} - \frac{1}{N}(\underline{n}^T\underline{a})\,\underline{m} \qquad (2)$$

$$\underline{b} = \frac{1}{N}\underline{n}^T\underline{\underline{C}} - \frac{1}{N}\left\{\underline{n}^T\left[\frac{1}{M}\underline{\underline{C}}\,\underline{m}^T - \frac{1}{M}(\underline{b}\,\underline{m}^T)\,\underline{n}\right\}\underline{m}\right.$$

$$\underline{b} = \frac{1}{N}\underline{n}^T\underline{\underline{C}} - \frac{1}{NM}(\underline{n}^T\underline{\underline{C}}\,\underline{m}^T)\,\underline{m} + \frac{1}{NM}(\underline{b}\,\underline{m}^T)\underbrace{\underline{n}^T\underline{n}}_{N}\,\underline{m}$$

$$\underline{b} = \frac{1}{N}\underline{n}^T\underline{\underline{C}} - \frac{1}{NM}(\underline{n}^T\underline{\underline{C}}\,\underline{m}^T)\,\underline{m} + \frac{1}{M}(\underline{b}\,\underline{m}^T)\,\underline{m}$$

$$\underline{b}\,m^T = \frac{1}{N}n^T\underline{\underline{C}}\,m^T - \frac{1}{N}(n^T C\, m^T) + (b\, m^T)$$

$$0 = 0$$

so equations 1 & 2 are singular. By inspection, a constant $c\,\underline{n}\,\underline{m}$ can be shifted between $a\underline{m}$ and $\underline{n}\,\underline{b}$. The singularity can be removed by adding a $c\,\underline{n}\,\underline{m}$ term and by imposing the constraints $\underline{n}^T\underline{a} = \underline{b}\,m = 0$.

ADDendum to MRN020. The normal equations are singular, because a constant can be moved between a_i and b_j. My solution is to extract

$$M_{ij} = c\,1_i 1_j + D_{ij}$$

The solution in an overall constant term
$$M_{ij} = c\,1_i 1_j + a_i 1_j + 1_i b_j + D_{ij}$$

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$$\sum_i \sum_j (M_{ij} - c\,1_i 1_j)^2 \Rightarrow \sum_i \sum_j 2\,1_i 1_j (M_{ij} - c\,1_i 1_j) = 0$$

$$\sum_i \sum_j M_{ij} = cNM \qquad c = \frac{1}{NM} \sum_i \sum_j M_{ij}$$ best fitting const matrix

$$M'_{ij} = M_{ij} - c\,1_i 1_j = M_{ij} - \frac{1}{NM}\Big(\sum_i \sum_j M_{ij}\Big) 1_i 1_j$$

$$\sum_i \sum_j M'_{ij} = \sum_i \sum_j M_{ij} - \sum_i \sum_j M_{ij} = 0$$

now write

$$M'_{ij} = a_i 1_j + 1_i b_j + c\,1_i 1_j \quad \text{with } c=0$$

$$\sum_i \sum_j M'_{ij} = 0 = M \sum_i a_i + N \sum_j b_j + 0 \qquad \text{now let } c = c_a + c_b = 0$$

$$M'_{ij} = (a_i + c_a 1_i) 1_j + 1_i (b_j + c_b 1_j) = a'_i 1_j + 1_i b'_j$$

now choose c_a so $\sum_i a'_i = 0$ $\quad \sum_i a_i + N c_a = 0 \quad c_a = -\frac{1}{N} \sum_i a_i$

$$c_b = +\frac{1}{N} \sum_i a_i$$

$$M'_{ij} = a'_i 1_j + 1_i b'_j$$

$$\sum_i \sum_j M'_{ij} = 0 + N \sum_j b'_j = 0 \qquad \text{so } b'_j = 0$$

Thus if we <u>choose</u> a c_a so $\sum_i a'_i = 0$, then $\sum_j b_j = 0$ also

in which case the normal eqns in MRN020 reduce to

$$a'_i = \frac{1}{M} \sum_j M'_{ij} \qquad b'_j = \frac{1}{N} \sum_i M'_{ij}$$

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% MRN 020 and 133
% best fit of a matrix to the sum of
%     a constant, a row vector, and a column vector
% M = c ones(N,1)*ones(1,M) + a*ones(1,M) + ones(N,1)*b + D

% random matrix
N=5;
M=4;
MM = random('normal',1,1,N,M);

% new way
L2MM = sum(sum(MM.^2));
c = sum(sum(MM))/(N*M);
MMmc = MM-c*ones(N,M);
ap = sum(MMmc,2)/M;
bp = sum(MMmc,1)/N;
MMr = c*ones(N,M) + ap*ones(1,M) + ones(N,1)*bp;
D = MM - MMr;
L2D = sum(sum(D.^2));

% old way
ax = zeros(N,1);
bx = zeros(1,M);
for i=[1:10]
    ax = (sum(MM,2)-sum(bx)*ones(N,1))/M;
    bx = (sum(MM,1)-sum(ax)*ones(1,M))/N;
end
MMrr = ax*ones(1,M) + ones(N,1)*bx;
DD = MM - MMrr;
L2DD = sum(sum(DD.^2));
L2X = sum(sum((MMr-MMrr).^2));

% check that the old and new ways yield the same answer
% (which they do, I've checked)
fprintf('norm M: %.2f  norm D1: %.2f norm  D2: %.2f  Norm (D1-D2): %.2f\n',
L2MM, L2D, L2DD, L2X);
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sample output:

norm M: 33.13 norm D1: 7.19 norm D2: 7.19 Norm (D1-D2): 0.00

norm M: 28.07 norm D1: 14.93 norm D2: 14.93 Norm (D1-D2): 0.00