Data Smoothing with Exponential Ch Maia May 2014

Consider (Ch)_{ij} = \varepsilon^2 \left( \frac{a}{2} \right) \exp(-a |i-j|)

and the data smoothing problem

\[
\begin{pmatrix}
    \frac{I}{Ch} & M \equiv \\
    0 & \frac{1}{Ch}
\end{pmatrix}\begin{pmatrix}
    d \\
    0
\end{pmatrix}
\]

1. The operator that is equivalent to $Ch^{-1}$ is $\varepsilon^2 \left( \frac{a^2}{1 + \varepsilon^2 \frac{d^2}{dx^2}} \right)$

2. The operator that is equivalent to $Ch^{-1/2}$ is $\varepsilon \left( 1 - a^{-1} \frac{d}{dx} \right)$
   (but note it is not self-adjoint)

3. The self-adjoint operator that is equivalent to $Ch^{-1/2}$ is $\varepsilon \left( 1 - a^{-1} \frac{d}{dx} \right)$

   but note it is not self-adjoint

4. In the continuum limit

   \[
   m(x) = (c \exp(-16\pi x)) \text{ convoluted with } d(x)
   \]

   with $c = (1 + \varepsilon^2)^{-1} \left( \frac{1}{2} \right)$ and $d = \frac{1 + \varepsilon^2}{\varepsilon} \cdot a$

5. Note autocorrelation $m$ not exponential

6. In order for autocorrelation $m$ to be exponential
   \[
   m = g \ast d \ \text{ with } g = \text{Re}(\alpha |x|)
   \]

7. But I haven't worked out $Ch^{-1/2}$ for this case
\[ A(v) = \frac{a}{2} e^{-a/2} = \frac{a}{2} f(x) \]
\[ \int_{-\infty}^{\infty} e^{-a-y} dy = 2 \int_{0}^{\infty} e^{-ax} dx = \frac{2}{a} e^{-ax} \bigg|_{0}^{\infty} = \frac{2}{a} \]

\[ A(k) = \frac{a}{2} \int_{-\infty}^{\infty} e^{-a/2} \cos{kx} \, dk = \frac{a^2}{k^2 + a^2} \]

\[ \frac{1}{A(k)} = \frac{k^2 + a^2}{a^2} = B(k) \]

\[ B(x) = \frac{1}{(2\pi)(a^2)} \int_{-\infty}^{\infty} (k^2 + a^2) \cos{kx} \, dk \]
\[ = \frac{1}{a^2} \left( \delta(x) + a^2 \delta(x) \right) \]
\[ B(x) = \frac{1}{a^2} \left( -\frac{d^2}{dx^2} + a^2 \right) = \frac{1}{a^2} \frac{d^2}{dx^2} \]

\[ A > 0 \]
\[ f = e^{a x} \]
\[ \frac{d}{dx} \left( e^{a x} \right) = \left( -a^2 e^{a x} + a^2 e^{a x} \right) = 0 \]
\[ A < 0 \]
\[ f = -e^{a x} \]
\[ \frac{d}{dx} \left( -e^{a x} \right) = \left( -a^2 e^{a x} - a^2 e^{a x} \right) = 0 \]

\[ x = 0 \]
\[ f = 0 \] so \( f = 0 \delta(0) \) and \( \frac{1}{a^2 \alpha} \frac{d^2}{dx^2} = \frac{ \delta(x) }{ \alpha(2) } \)

**Upshot** operator that inverts convolution by
\[ A(x) = \frac{a}{2} e^{a x} \]

Is
\[ 1 - \frac{1}{a^2} \frac{d^2}{dx^2} \]
\[ (G^T C_d^{-1} G + H^T C_h^{-1} H) m = G^T C_d^{-1} d + 0 \]

\[ G = I \quad H = I \quad C_h = \varepsilon^2 A \quad A_{ij} = \frac{a}{2} e^{-a|x_i - x_j|} \]

\[ C_d = I \]

\[ (\varepsilon^2 A^{-1} + I) m = d \]

\[ \text{let } B = A^{-1} \]

now take continuum limit

\[ \varepsilon^2 B \times m + m = d \]

\[ -\varepsilon^2 (m - a^2 \frac{d^2 m}{d x^2}) + m = d \]

\[ -\varepsilon^2 a^2 \frac{d^2 m}{d x^2} + (1 + \varepsilon^2) m = d \quad d = 8 \text{ to get G,F} \]

\[ \frac{d^2 m}{d x^2} - a^2 \frac{(1+\varepsilon^2)}{\varepsilon^2} m = -\delta(x) \frac{a^2}{\varepsilon^2} \]

\[ b = \frac{(1+\varepsilon^2)^{1/2}}{\varepsilon} \]

\[ \frac{b}{b} = \frac{a}{a} = \frac{\varepsilon}{(1+\varepsilon^2)^{1/2}} \]

\[ \text{output scale length} \]

\[ \text{input scale length} \]

\[ m = c e^{-b/|x|} \]

\[ x < 0 \quad m = c e^{-b x} \]

\[ \frac{d m}{dx} = c b e^{-b x} \]

\[ \frac{d^2 m}{d x^2} = -2 b e^{-b x} \]

\[ x > 0 \quad m = c e^{b x} \]

\[ \frac{d m}{dx} = c b e^{b x} \]

\[ \frac{d^2 m}{d x^2} = 2 b e^{b x} \]

\[ m_{x=0} = \frac{a}{b} \]

\[ b = \frac{a}{a^2} \frac{1}{2} e - b/|x| \]

\[ m = (1+\varepsilon)^{1/2} e - b/|x| \]

\[ m = \frac{a^2}{b^2} \frac{1}{2} e^{b x} \]

\[ a \frac{b}{b} = a^{-1} \]

\[ \frac{a}{b} = \frac{a}{a} \]

\[ a \frac{b}{b} = (1+\varepsilon)^{1/2} b \]

\[ \text{as } \epsilon \to 0 \]
If \( m = f \star d \) then \( a_m(x) = f(-t) * d(-t) = f(t) * d(t) = (f \star f) * a_d \) so if \( a_d = \delta \) \( a_m = f \star f \) is symmetric.

A function has autocorrelation \( \text{cor} \) \( e^{-x^2/2} \).

What is the function, \( f(x) \)?

\[(f \star f)(x) = a(x) = f \star f(-x)\]

Fourier transform

\[a(k) = \mathcal{F}(f)(k) f(k) = f(k)^2\]

\[a(k) = 2 \int_0^\infty e^{-x^2/2} \cos(kx) \, dx\]

6.13.893.2

\[\int_0^\infty e^{-x^2/2} \cos(kx) \, dx = \frac{\sqrt{2\pi \beta}}{\beta^2 + k^2}\]

\[a(k) = 2 \frac{\alpha}{k^2 + \alpha^2} \quad f(k) = \frac{\sqrt{12\pi \beta}}{\sqrt{k^2 + \alpha^2}}\]

6.13.754.2

\[\int_0^\infty \cos(ax) \, dx = K_0(x)\]

\[f(x) = \frac{\sqrt{2\pi \beta}}{\beta + \alpha^2}\]

\[K_0(x) = K_0(\sqrt{\alpha^2 + \beta^2})\]

6.11.671.14

\[\int_0^\infty K_0(\beta x) \cos(ax) \, dx = \frac{1}{2 \sqrt{x^2 + \beta^2}}\]

\[\int_0^\infty K_0(ax) \cos(kx) \, dx = \frac{1}{2 \sqrt{k^2 + \alpha^2}}\]
\[
\begin{aligned}
\left( A_1 - \frac{1}{s^2} \right) u(r_L) &= -f(r_L) \\
A_1 &= \nabla \cdot \nabla_{1} = \nabla^2 - \frac{B}{r} \cdot \nabla = \nabla^2 - \frac{\hat{r}}{r} \\
G(r_L) &= \frac{1}{20} K_0 \left( \frac{r_L}{5} \right) \quad \text{for} \quad \left( k_{\perp}^2 + \frac{1}{s^2} \right) G(k_{\perp}) = 1 \\
\Delta_{L} &= \nabla \cdot \nabla - \nabla \cdot (\hat{r} \cdot \nabla) = 2D \nabla^2 - \frac{1}{r} \frac{\partial}{\partial r} \frac{2}{s} \\
\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) &= \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \\
\Delta_1 u &= \frac{1}{s^2} u \\
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{d u}{dr} &= s^2 u \\
x^2 u &= \frac{d^2 u}{dy^2} + x \frac{d}{dx} - (x^2 + x^2) y = 0 \\
\end{aligned}
\]
\[ H^T C^{-1}_n H = H^T A^T A^{-1}_n H = (A H)^T (A H) \]

so one must factorize \( C^{-1}_n \) as \( A^T A \).

Switching to operator, one must factorize
\[ C^{-1}_n = 1 - \alpha^{-2} \frac{d^2}{dx^2} \]
so that \( C^{-1}_n = A^T A \).

Try \( A = (1 + \alpha^{-1} \frac{d}{dx}) \) \( A^T = (1 - \alpha^{-1} \frac{d}{dx}) \)

\[ A^T A = (1 - \alpha^{-1} \frac{d}{dx}) (1 + \alpha^{-1} \frac{d}{dx}) = 1 - \alpha^{-2} \frac{d^2}{dx^2} \]

but \( A \) is not self-adjoint. However, for matrices, we could symmetrize \( A \) by a unitary transformation \( U \) such that \( U^T A U = \frac{1}{2} (A + A^T) \).

So we need a linear operator that is symmetric in the same sense. Try the Hilbert transform \( i \mathcal{H} \), since \( i \mathcal{H} (i \mathcal{H} m) = (i) (\mathcal{H} m) = m \)

\[ i \mathcal{H} (1 + \alpha^{-1} \frac{d}{dx}) = -i \alpha^{-1} \mathcal{H} (\frac{d}{dx}) \] (Since \( \mathcal{H} (\text{const}) = 0 \)

\[ \mathcal{H} (u) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{u(t)}{t-x} dt \] (Wikipedia "Hilbert Transform")

\[ \mathcal{H} (\frac{du}{dx}) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{u(t)}{(x-t)^2} dt \]

but dubious whether convolution by \( \frac{1}{\pi} \frac{x^2}{x^2} \)
is well-defined. It seems to diverge.

Note \( \mathcal{H} (\frac{du}{dx}) = \frac{1}{\alpha} \mathcal{H} (u) \). The finite difference derivative of

The DHT

\[ \left\{ \begin{array}{ll}
\frac{1}{2n} & \text{odd} \\
0 & \text{even}
\end{array} \right. \]

except for odd-even chatter.

The matrix \( B = I - \alpha^2 D_2 \) where \( D_2 \) = second difference does well in satisfying \( C = B^{-1} \) and The matrix \( A = I - \alpha^2 D_1 \) does well in satisfying \( C = (A^T A)^{-1} \), for 100x100 matrices, except for edge effects.