Some work I did on differential operators related to the data smoothing problem.

(1) operator that inverts $\dot{g}$ where $g = \frac{a}{2}\exp\{-a|x|\}$

(2) operator that inverts $(\delta - g)*$

(2.5) soln of $\varepsilon^2(\delta-g)*(\delta-g)*m + m = \delta$

(6) soln of $-\varepsilon^2 \frac{d^2}{dx^2} m + m = f(x)$

(7) cross-correlation $g \star g$

Late May, 2014, B. Menke with discussion w/ Zach Bilon operator That inverts $\dot{g}$ is $\left(-a\,\mathrm{sgn}(x) - \frac{1}{a^2}\frac{d}{dx}\right)$

$$g(x) = \frac{a}{2}e^{-a|x|}$$

$$\dot{g}(x) = -\frac{a^2}{2}\,\mathrm{sgn}(x)\,e^{-a|x|}$$

$$\ddot{g}(x) = -a^2\delta(x)\,e^{-a|x|} + \frac{a^3}{2}e^{-a|x|}$$

$$\left(1 - \frac{1}{a^2}\frac{d^2}{dx^2}\right)g(x) = g(x) - \frac{1}{a^2}\ddot{g}(x)$$

$$= \frac{a}{2}e^{-a|x|} + \left(\frac{1}{a^2}\right)(a^2)\,\delta(x)\,e^{-a|x|} - \frac{1}{a^2}\frac{a^3}{2}e^{-a|x|}$$

$$= \delta(x) \quad ✓$$

$$\left(-a\,\mathrm{sgn}(x) - \frac{1}{a^2}\frac{d}{dx}\right)\dot{g} =$$

$$(-a)\,\mathrm{sgn}(x)\left(-\frac{a^2}{2}\right)\mathrm{sgn}(x)\,e^{-a|x|} \longmapsto \frac{a}{2}e^{-a|x|}$$

$$\left(-\frac{1}{a^2}\right)(-a^2)\,\delta(x)\,e^{-a|x|} \longmapsto \delta(x)$$

$$\left(-\frac{1}{a^2}\right)\left(+\frac{a^3}{2}\right)e^{-a|x|} \longmapsto -\frac{a}{2}e^{-a|x|}$$

$$= \delta(x)$$

Operator that inverts $(\delta - g)*$ is $\left(-\frac{a^2}{2}|x| + \delta(x)\right)*$ ②

$$m - g * m = \delta$$

$$\mathcal{L} = 1 - a^{-2}\frac{d}{dx^2} \quad \text{inverts } g*$$

$$\mathcal{L} m - m = \mathcal{L}\delta$$

$$\left(1 - a^{-2}\frac{d}{dx^2}\right)m - m = \left(1 - a^{-2}\frac{d}{dx^2}\right)\delta$$

$$-a^{-2}\frac{d}{dx^2} m = \delta - a^{-2}\ddot{\delta}$$

try $m = -\frac{a^2}{2}|x| + \delta(x) + C \qquad C = \text{arb.}$

$$\frac{dm}{dx} = -\frac{a^2}{2}\,\mathrm{sgn}(x) + \dot{\delta}(x)$$

$$\frac{d^2m}{dx^2} = -a^2\delta(x) + \ddot{\delta}(x)$$

$$-a^{-2}\frac{d^2m}{dx^2} = \delta(x) - a^{-2}\ddot{\delta}(x) \quad \checkmark$$

$=$ check $m = \frac{-a^2}{2}|x| + \delta(x)$

$$-\frac{a^2}{2}|x| + \delta(x) + \frac{a^2}{2}|x| * g - g$$

would be correct if $\frac{a^2}{2}|x| * g = g + \frac{a^2}{2}|x|$

check Fourier Transform $|x| \to \frac{-2}{k^2}$, $g \to \frac{a^2}{k^2+a^2}$

$$\frac{a^2}{2}\frac{(-2)}{k^2}\frac{a^2}{a^2+k^2} \stackrel{?}{=} \frac{a^2}{k^2+a^2} - \frac{a^2}{2}\frac{2}{k^2}$$

$$= \frac{k^2a^2}{k^2(k^2+a^2)} - \frac{a^2}{2}\frac{2(a^2+k^2)}{k^2(a^2+k^2)} \stackrel{\checkmark}{=} \frac{a^4}{k^2(a^2+k^2)}$$

checks

$$\varepsilon^2(\delta-g)*(\delta-g)*m \quad + m = \delta$$

$$\varepsilon^2 m - 2\varepsilon^2 g*m + \varepsilon^2 g*g*m + m = \delta$$

$$(1+\varepsilon^2)m - 2\varepsilon^2 g*m + \varepsilon^2 g*g*m = \delta$$

$$(1+\varepsilon^2) L_g L_g m - 2\varepsilon^2 L_g m + \varepsilon^2 m = L_g L_g \delta$$

$$L_g = (1 - a^{-2}\frac{d^2}{dx^2}) \qquad \rightarrow 1 + a^{-2}k^2$$

$$L_g L_g = 1 - 2a^{-2}\frac{d^2}{dx^2} + a^{-4}\frac{d^4}{dx^4} \qquad \rightarrow (1+a^{-2}k^2)^2$$

$$(1+\varepsilon^2)\left(m - 2a^{-2}\frac{d^2m}{dx^2} + a^{-4}\frac{d^4m}{dx^4}\right) - 2\varepsilon^2\left(m - a^{-2}\frac{d^2m}{dx^2}\right) + \varepsilon^2 m = L_g L_g \delta$$

$$-2 - 2\varepsilon^2 + 2\varepsilon^2 = -2$$

$$(1+\varepsilon^2)a^{-4}\frac{d^4m}{dx^4} + \left[-(1+\varepsilon^2)2a^{-2} + 2\varepsilon^2 a^{-2}\right]\frac{d^2m}{dx^2} + \left[(1+\varepsilon^2) - 2\varepsilon^2 + \varepsilon^2\right] = L_g L_g \delta$$

$$(1+\varepsilon^2)a^{-4}\frac{d^4m}{dx^4} + 2a^{-2}\frac{d^2m}{dm^2} + m = L_g L_g \delta$$

$$\left((1+\varepsilon^2)k^4a^{-4} + 2k^2a^{-2} + 1\right)m(k) = (1+a^{-2}k^2)^2$$

$$\left[(1+a^{-2}k^2)^2 + \varepsilon^2 a^{-4}k^4\right]$$

$$m(k) = \frac{(1+a^{-2}k^2)^2}{(1+a^{-2}k^2)^2 + \varepsilon^2 a^{-4}k^4} = 1 - \frac{\varepsilon^2 a^{-4}k^4}{(1+a^{-2}k^2)^2 + \varepsilon^2 a^{-4}k^4}$$

$$m = e^{-bx} \qquad \frac{d^2m}{dx^2} = b^2 e^{-bx} \qquad \frac{d^4m}{dx^4} = b^4 e^{-bx}$$

$$\left[(1+\varepsilon^2)a^{-4}b^4 - 2a^{-2}b^2 + 1\right]e^{-bx}$$

$$(1 - a^{-2}b^2)^2 + \varepsilon^2 a^{-4}b^4 \qquad \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$b^2 = \frac{2a^{-2} \pm \left[4a^{-4} - 4(1+\varepsilon^2)a^{-4}\right]^{1/2}}{2} = \frac{2a^{-2} \pm 2a^{-2}\varepsilon i}{2} = a^{-2}(1 \pm i\varepsilon)$$

$$b = \pm a^{-1}(1 \pm i\varepsilon)^{1/2} \simeq \pm a^{-1}(1 \pm \tfrac{1}{2}\varepsilon)$$

$$\left(1 + (1+i\varepsilon)a^{-2}k^2\right)\left(1 + (1-i\varepsilon)a^{-2}k^2\right)$$

$$1 + (1+i\varepsilon)a^{-2}k^2 + (1-i\varepsilon)a^{-2}k^2 + (1+i\varepsilon)(1-i\varepsilon)a^{-4}k^4$$

$$1 + 2a^{-2}k^2 + (1+\varepsilon^2)a^{-4}k^4$$

see GR 3.728.1

$$m(k) = \frac{(1+a^{-2}k^2)^2}{\left(1+(1+i\varepsilon)a^{-2}k^2\right)\left(1+(1-i\varepsilon)a^{-2}k^2\right)} \qquad \begin{matrix}\beta^2 \\ \gamma^2\end{matrix} = \frac{a^2 (1 \mp i\varepsilon)}{(1 \pm i\varepsilon)(1 \mp i\varepsilon)} = \frac{a^2(1 \mp i\varepsilon)}{1+\varepsilon^2}$$

$$\rightarrow \beta^2 - \gamma^2 = \frac{a^2}{1+\varepsilon^2}(1 - 2i\varepsilon)$$

$$(1+\varepsilon^2)a^{-4}\frac{d^4 m}{dx^2} - 2a^{-2}\frac{d^2m}{dx^2} + 1 \xrightarrow{FT}$$

$$(1+\varepsilon^2)a^{-4}k^4 + 2a^{-2}k^2 + 1 =$$

$$(1+(1+i\varepsilon)a^{-2}k^2)(1+(1-i\varepsilon)a^{-2}k^2) =$$

$$(1+i\varepsilon)(1-i\varepsilon)a^{-4}k^2 + (1+i\varepsilon)a^{-2}k^2 + (1-i\varepsilon)a^{-2}k^2 + 1$$

$$(1+\varepsilon^2)a^{-4}k^2 + 2a^{-2}k^2 + 1 \quad \checkmark$$

$$(1+(1+i\varepsilon)a^{-2}k^2) \quad (1+(1-i\varepsilon)a^{-2}k^2)$$

$$(1+i\varepsilon)a^{-2}\left((1+i\varepsilon)^{-1}a^2+k^2\right) \quad (1-i\varepsilon)a^{-2}\left((1-i\varepsilon)^{-1}a^2+k^2\right)$$

$$(1+\varepsilon^2)a^{-4}(\beta^2+k^2)(\gamma^2+k^2)$$

$$\beta^2 = \frac{a^2}{(1+i\varepsilon)} = \frac{(1-i\varepsilon)a^2}{(1+i\varepsilon)(1-i\varepsilon)} = \frac{(1-i\varepsilon)a^2}{(1+\varepsilon^2)}$$

$$\gamma^2 = \frac{a^2}{(1-i\varepsilon)} = \frac{(1+i\varepsilon)a^2}{(1+i\varepsilon)(1-i\varepsilon)} = \frac{(1+i\varepsilon)a^2}{(1+\varepsilon^2)}$$

$$\beta = \frac{a}{r}(p-iq) \qquad \gamma = \frac{a}{r}(p+iq)$$

$$\beta^2 - \gamma^2 = \frac{(1-i\varepsilon)a^2}{(1+\varepsilon^2)} - \frac{(1+i\varepsilon)a^2}{(1+\varepsilon^2)} = \frac{-2i\varepsilon a^2}{(1+\varepsilon^2)}$$

purely imag

$$\beta\gamma = \frac{a^2}{r^2}(p^2+q^2) = \frac{a^2}{r^2}(1+\varepsilon^2)^{1/2}$$

purely real

wikipedia

$$(a+bi)^{1/2} = p+qi$$

$$p = \left(\frac{a+(a^2+b^2)^{1/2}}{2}\right)^{1/2}$$

$$q = \mathrm{sgn}(b)\left(\frac{-a+(a^2+b^2)^{1/2}}{2}\right)^{1/2}$$

$$(1\pm i\varepsilon)^{1/2} = p \pm iq$$

$$p = \left(\frac{1+(1+\varepsilon^2)^{1/2}}{2}\right)^{1/2}$$

$$q = \left(\frac{-1+(1+\varepsilon^2)^{1/2}}{2}\right)^{1/2}$$

$$r = (1+\varepsilon^2)^{1/2} \quad p = \left(\frac{1+r}{2}\right)^{1/2}$$

$$q = \pm\left(\frac{-1+r}{2}\right)^{1/2}$$

note $\varepsilon \to \infty$

$p = q = \varepsilon^{1/4}$

$$\beta e^{-\gamma x} - \gamma e^{-\beta x} =$$

$$\frac{a}{r}(p-iq)e^{-\frac{a}{r}(p+iq)x} - \frac{a}{r}(p+iq)e^{-\frac{a}{r}(p-iq)x} =$$

$$\frac{a}{r}e^{-\frac{a}{r}px}\left[(p-iq)e^{-i\frac{a}{r}qx} - (p+iq)e^{+i\frac{a}{r}qx}\right]$$

$$A^* e^{-iB} - A e^{+iB}$$

$$(A_r - iA_i)(\cos B - i\sin B) - (A_r + iA_i)(\cos B + i\sin B) =$$

$$A_r\cos B - iA_r\sin B - iA_i\cos B - A_i\sin B$$

$$- A_r\cos B - iA_r\sin B - iA_i\cos B + A_i\sin B =$$

$$2i(-A_r\sin B - A_i\cos B)$$

$$-2i\left(p\sin\frac{aq}{r}x + q\cos\frac{aq}{r}x\right)$$

f: $pES + qEC$

$\frac{df}{dx}$: $-p^2 HES + pq HEC - pq HEC - q^2 HES$

$-(p^2+q^2) HES$

$\frac{d^2f}{dx^2}$: $-\cancel{(p^2+q^2)}\delta ES + p(p^2+q^2)ES + q(p^2+q^2)EC$

$\frac{d^3f}{dx^3}$: $-\cancel{(p^2+q^2)}\dot{\delta} - p^2(p^2+q^2)HES + pq(p^2+q^2)HEC$
$+ pq(p^2+q^2)HEC + q^2(p^2+q^2)HES$

$-\cancel{(p^2+q^2)}\dot{\delta} - (p^2-q^2)(p^2+q^2)HES + 2pq(p^2+q^2)HEC$

(p^4-q^4)

$\frac{d^4f}{dx^4}$: $-\cancel{(p^2+q^2)}\ddot{\delta} - (p^2-q^2)(p^2+q^2)\delta + \cancel{p(p^2+q^2)^2ES - q(p^2+q^2)^2EC}$

$4pq(p^2+q^2)\delta$

$\cancel{(p^2-q^2)(p^2+q^2)}$

$+p(p^4-q^4)ES - q(p^4-q^4)EC$

$-\cancel{(p^2+q^2)}\delta'' - 2p^2q(p^2+q^2)EC - 2pq^2(p^2+q^2)ES$

$-\cancel{(p^2+q^2)}\delta'' + \left[p(p^4-q^4) - 2pq^2(p^2+q^2)\right]ES$
$\left[-q(p^4-q^4) - 2p^2q(p^2+q^2)\right]EC$

$$C_h^{-1/2} \to \varepsilon \frac{d}{dx}$$

$$C_h^{-1} \to \varepsilon^2 \left(\frac{d}{dx}\right)^+ \left(\frac{d}{dx}\right) = -\varepsilon^2 \frac{d^3}{dx^2}$$

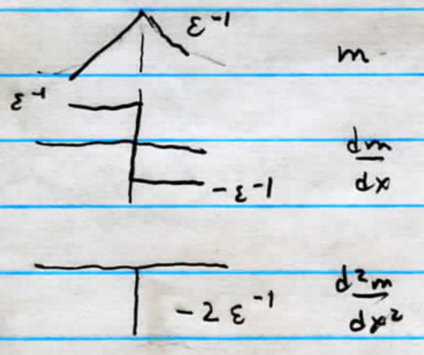
$$\left(-\varepsilon^2 \frac{d^2}{dx^2} + 1\right) m = \delta(x)$$

$$\varepsilon^2 \frac{d^2 m}{dx^2} - m = -\delta(x)$$

try $m = \frac{\varepsilon^{-1}}{2} e^{-\varepsilon^{-1}|x|}$

$x < 0$; $m = \frac{\varepsilon^{-1}}{2} e^{\varepsilon^{-1} x}$ $\quad \frac{\varepsilon^{-1}}{2} \varepsilon^2 \varepsilon^{-2} e^{\varepsilon^{-1} x} - \frac{\varepsilon^{-1}}{2} e^{\varepsilon^{-1} x} = 0$ ✓

$x > 0$; $m = \frac{\varepsilon^{-1}}{2} e^{-\varepsilon^{-1} x}$ $\quad \frac{\varepsilon^{-1}}{2} \varepsilon^2 \varepsilon^{-2} e^{\varepsilon^{-1} x} - \frac{\varepsilon^{-1}}{2} e^{\varepsilon^{-1} x} = 0$ ✓



$$\varepsilon^2 \frac{\varepsilon^{-1}}{2} \left(-2\varepsilon^{-1} \delta(x)\right) = -\delta(x) \checkmark$$

so G.F. is

$$g(x) = \frac{1}{2\varepsilon} \exp(-|x|/\varepsilon)$$

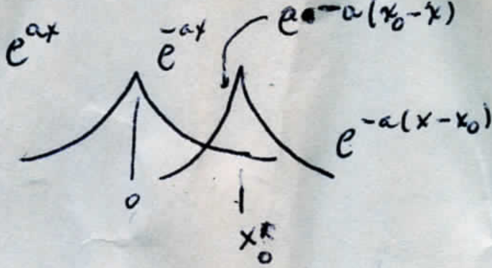
operator that inverts $-\varepsilon^2 \frac{d^2}{dx^2}$ is $f(x) *$ with

$$f = \frac{1}{2\varepsilon}\left(1 - \frac{|x|}{\varepsilon}\right) *$$

since it solves $-\varepsilon^2 \frac{d^2 f}{dx^2} = \delta(x)$

$$(-\varepsilon^2)\left(\tfrac{1}{2}\varepsilon^{-1}\right)(-2\varepsilon^{-1})\delta(x) = \delta(x) \checkmark$$

I checked this last result numerically. OK, except for edge effects.



L: $e^{ax}e^{ax}e^{-ax_0} = e^{2ax}e^{-ax_0}$

$$\int_{-\infty}^{0} e^{2ax}\,dx = \frac{1}{2a}e^{2ax}\Big|_{-\infty}^{0} = \frac{1}{2a} \Rightarrow \frac{1}{2a}e^{-ax_0}$$

R: $e^{-ax}e^{-ax}e^{ax_0} = e^{-2ax}e^{ax_0}$

$$\int_{x_0}^{\infty} e^{-2ax}\,dx = -\frac{1}{2a}e^{-2ax}\Big|_{x_0}^{\infty} = +\frac{1}{2a}e^{-2ax_0} \Rightarrow +\frac{1}{2a}e^{-ax_0}$$

M: $e^{-ax}e^{ax}e^{-ax_0}$

$$\int_{0}^{x_0} dx = x_0 \to |x_0|e^{-a|x_0|} + \frac{1}{a}e^{-a|x_0|}$$

$$\left(|x_0| + a^{-1}\right)e^{-a|x_0|}$$