Relationship between the Fourier Transform of a Time Series and the Fourier Transforms of its Right and Left Halves

Bill Menke, July 25, 2015

Statement if the problem: We consider a time series **a** of length N

$$a_j \quad (0 \le j \le N-1)$$

where N is even. This time series has Discrete Fourier Transform (DFT) $\tilde{\mathbf{a}}$, defined as:

$$\tilde{a}_k = \sum_{j=0}^{N-1} a_j \exp\{-i\, jk\, (2\pi/N)\} \quad (0 \le k \le N-1)$$

Now suppose that the time series is subdivided into two pieces, **b**, and **c**, of equal equal length $M = N/2$. What is the relationship between the DFT's of the two pieces, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$, and the DFT of the original time series **a**?

Sub-sampling in frequency. While all the time series have the same Nyquist frequency, the frequency-spacing of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$ is twice that of $\tilde{\mathbf{a}}$. We first develop a technique to sub-sample $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$ by a factor of two, which is to say, to evaluate $\tilde{d}_k \equiv \tilde{b}_{k+½}$ and $\tilde{e}_k \equiv \tilde{c}_{k+½}$ $(0 \le k \le M-1)$. From the definition of the DFT, we find:

$$\tilde{d}_k \equiv \tilde{b}_{k+½} = \sum_{j=0}^{M-1} b_j \exp\{-i\, j\, (k+½)\, (2\pi/N)\}$$

$$= \sum_{j=0}^{M-1} b_j \exp\{-i\, j\, (\pi/N)\} \exp\{-i\, jk\, (2\pi/N)\}$$

and similarly for $\tilde{e}_k \equiv \tilde{c}_{k+½}$. The half-integer values are obtained by taking the DFT of $[x_j b_j]$ and $[x_j c_j]$,where $x_j = \exp\{-i\, j\, (\pi/N)\}$. The convolution theorem implies that $\tilde{\mathbf{d}} = \tilde{\mathbf{x}} * \tilde{\mathbf{b}}/M$ and $\tilde{\mathbf{e}} = \tilde{\mathbf{x}} * \tilde{\mathbf{c}}/M$; that is, the elements of $\tilde{\mathbf{d}}$ and $\tilde{\mathbf{e}}$ are linear combinations of the elements of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$, respectively. As expected, the half-integer values provide no new information; they can be viewed as an interpolation of the whole-integer values, with a kernel of $\tilde{\mathbf{x}}$.

The filter $\tilde{\mathbf{x}}$ can be determined analytically, starting with the DFT of $\mathbf{x}$ and applying equations1.341.1 and 1.341.3 of Gradshteyn & Ryzhik (1980) :

$$\tilde{x}_k = \sum_{j=0}^{M-1} \exp\{-i\, j\, (\pi/N)\} \exp\{-i\, jk\, (2\pi/N)\} =$$

$$\sum_{j=0}^{M-1} \exp\{-i\, j(k + ½)(2\pi/M)\} = A - iB$$

$$A = \sum_{j=0}^{M-1} \cos\{i\, j(k + ½)(2\pi/M)\} =$$

$$\cos\left(\pi(k + ½)\frac{(M-1)}{M}\right)\frac{\sin\big(\pi(k + ½)\big)}{\sin\left(\frac{\pi(k + ½)}{\mathrm{M}}\right)} =$$

$$\left\{\cos\big(\pi(k + ½)\big)\cos\left(\frac{\pi(k + ½)}{M}\right) + \sin\big(\pi(k + ½)\big)\sin\left(\frac{\pi(k + ½)}{M}\right)\right\} \times$$

$$\frac{\sin\big(\pi(k + ½)\big)}{\sin\left(\frac{\pi(k + ½)}{\mathrm{M}}\right)} = 1$$

$$B = \sum_{j=0}^{M-1} \sin\{i\, j(k + ½)(2\pi/M)\} =$$

$$\sin\left(\pi(k + ½)\frac{(M-1)}{M}\right)\frac{\sin\big(\pi(k + ½)\big)}{\sin\left(\frac{\pi(k + ½)}{\mathrm{M}}\right)} =$$

$$\left\{\sin\big(\pi(k + ½)\big)\cos\left(\frac{\pi(k + ½)}{M}\right) - \cos\big(\pi(k + ½)\big)\sin\left(\frac{\pi(k + ½)}{M}\right)\right\} \times$$

$$\frac{\sin\left(\pi(k+½)\right)}{\sin\left(\frac{\pi(k+½)}{M}\right)} = \cot\left(\frac{\pi(k+½)}{M}\right)$$

since $\sin^2\left(\pi(k+½)\right)=1$ and $\cos\left(\pi(k+½)\right)=0$ when k is an integer. These formulas have been checked by numerical evaluation of test cases. Note that the real part of $\tilde{\mathbf{x}}$ is a constant and that the imaginary part has a shape similar to that of the $1/k$ filter associated with the Hilbert transform (Figure 1).

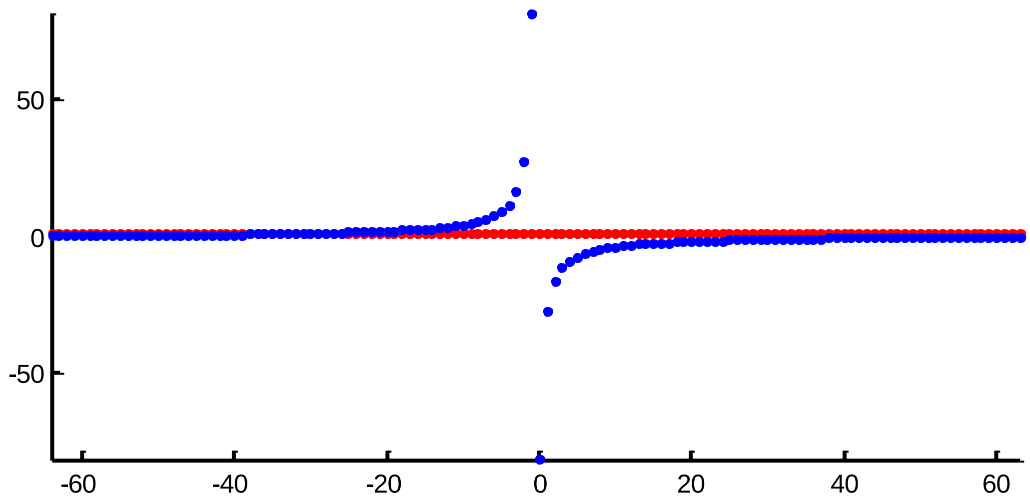

Figure 1. The real (red) and imaginary (blue) parts of the filter $\tilde{\mathbf{x}}$ in the $N=256$ case.

Constructing $\tilde{\mathbf{a}}$.from $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$. Let $\mathbf{z}$ be a time series consisting of M zeros. Then:

$$\mathbf{a}=\mathbf{f}+\mathbf{g} \quad \text{with} \quad \mathbf{f}=[\mathbf{b};\mathbf{z}] \quad \text{and} \quad \mathbf{g}=[\mathbf{z};\mathbf{c}]$$

Here, the semicolon represents concatenation; e.g. $\mathbf{f}$ is a length-N time series consisting of the M elements of $\mathbf{b}$ followed by M zeros. The $\mathbf{f}$ and $\mathbf{g}$ time series are equivalent to windowed versions of $\mathbf{a}$ obtained by multiplying its left and right halves, respectively, by zeros. Because of the linearity of the DFT:

$$\tilde{\mathbf{a}}=\tilde{\mathbf{f}}+\tilde{\mathbf{g}}$$

We now define a length-N vector $\tilde{\mathbf{u}}$ consisting of interleaved elements of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{d}}$ that represents $\tilde{\mathbf{b}}$ interpolated to half its normal frequency spacing; that is $\tilde{\mathbf{u}} = [\mathrm{b}_0, \mathrm{d}_0, \mathrm{b}_1, \mathrm{d}_1, \mathrm{b}_2, \mathrm{d}_2, \cdots]$. The zeros at the end of $\mathbf{f}$ have no affect on its DFT, so $\tilde{\mathbf{f}} = \tilde{\mathbf{u}}$. In contrast, $\tilde{\mathbf{g}}$ is not equal to $[\mathrm{c}_0, \mathrm{e}_0, \mathrm{c}_1, \mathrm{e}_1, \mathrm{c}_2, \mathrm{e}_2, \cdots]$, because $\mathbf{g}$ is right-shifted by M samples with respect to $\mathbf{c}$ by the zeros at it start. In the frequency domain, the shift is equivalent to multiplication by:

$$\tilde{y}_k = \exp\big(-iMk(2\pi/N)\big) = \cos(i\pi k) - i\sin(i\pi k)$$

which is just the sequence $\tilde{\mathbf{y}} = [1, -1, 1, -1, \dots]^{\mathrm{T}}$. Thus, $\tilde{\mathbf{g}} = \tilde{\mathbf{v}}$ with $\mathbf{v} = [\mathrm{c}_0, -\mathrm{e}_0, \mathrm{c}_1, -\mathrm{e}_1, \mathrm{c}_2, -\mathrm{e}_2, \cdots]$.

An example of the reconstruction process is shown in Figures 2 and 3.

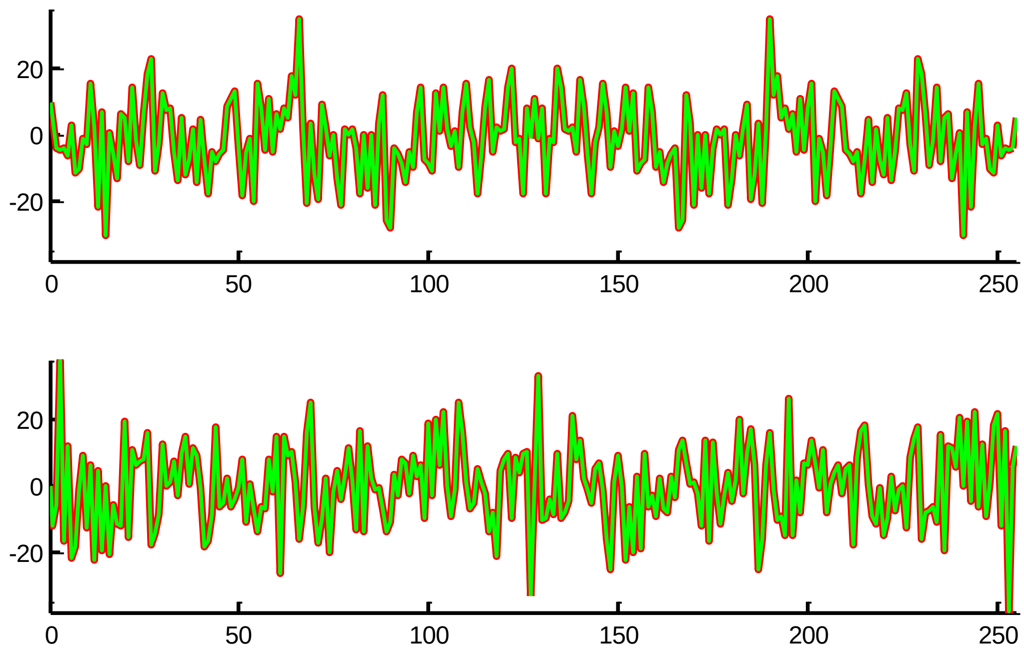

Figure 2. The real (top) and imaginary (bottom) parts of the DFT of exemplary time series $\mathbf{a}$ in the $N = 256$ case. Both $\tilde{\mathbf{a}}$ (red) and $\tilde{\mathbf{f}} + \tilde{\mathbf{g}}$ (green) are shown.

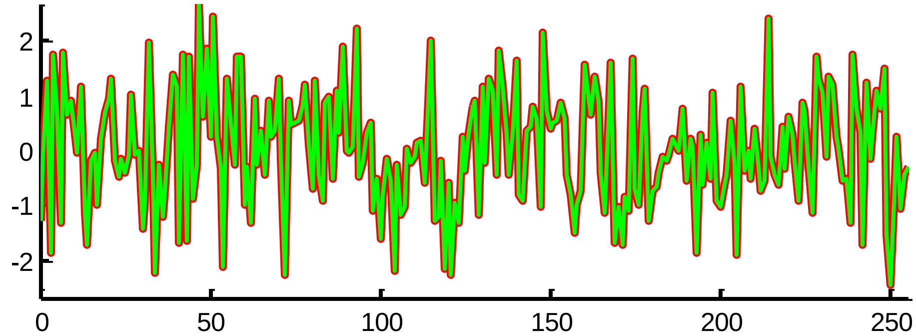

Figure 2. The exemplary time series $\mathbf{a}$ in the $N = 256$ case. Both $\mathbf{a}$ (red) and the inverse DFT of $\tilde{\mathbf{f}} + \tilde{\mathbf{g}}$ (green) are shown.

Discussion. This study was motivated by the practice of estimating the power spectra of a long time series by averaging the spectra of shorter pieces of it (e.g. Welch, 1967). Intuitively, one would expect that average over two adjacent frequencies of $\tilde{\mathbf{a}}$ would be more or less equivalent to averaging the corresponding single frequency of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$:

$$S^{intuitive} = |\tilde{\mathbf{a}}_{2j}|^2 + |\tilde{\mathbf{a}}_{2j+1}|^2 \approx 2|\tilde{\mathbf{b}}_j|^2 + 2|\tilde{\mathbf{c}}_j|^2 = X^{intuitive}$$

(The factor of two has been added to compensate for the power in the short time series being half that in the long one). The exact result is:

$$S = |\tilde{\mathbf{a}}_{2j}|^2 + |\tilde{\mathbf{a}}_{2j+1}|^2 = |\tilde{\mathbf{b}}_j + \tilde{\mathbf{c}}_j|^2 + |\tilde{\mathbf{d}}_j - \tilde{\mathbf{e}}_j|^2 = X + Y$$

with

$$X = |\tilde{\mathbf{b}}_j|^2 + |\tilde{\mathbf{d}}_j|^2 + |\tilde{\mathbf{c}}_j|^2 + |\tilde{\mathbf{e}}_j|^2 \text{ and } Y = \tilde{\mathbf{b}}_j^*\tilde{\mathbf{c}}_j + \tilde{\mathbf{b}}_j\tilde{\mathbf{c}}_j^* - \tilde{\mathbf{d}}_j^*\tilde{\mathbf{e}}_j - \tilde{\mathbf{d}}_j\tilde{\mathbf{e}}_j^*$$

The intuitive result differs from the exact one in two important respects. First, the term Y creates fluctuations in S not present in $S^{intuitive}$. Second, $X^{intuitive} \neq X$ since $\tilde{\mathbf{d}}_j \neq \tilde{\mathbf{b}}_j$ and $\tilde{\mathbf{e}}_j \neq \mathbf{c}_j$. For smooth spectra, one might expect $\tilde{\mathbf{d}}_j \approx \tilde{\mathbf{b}}_j$ and $\tilde{\mathbf{e}}_j \approx \mathbf{c}_j$, since $\tilde{\mathbf{d}}_j$ and $\tilde{\mathbf{e}}_j$ are the interpolated vales of $\tilde{\mathbf{b}}_{j+½}$ and $\tilde{\mathbf{c}}_{j+½}$, in which case $S^{intuitive} \approx S$. However, for rapidly-varying spectra, $\tilde{\mathbf{d}}_j$ and $\tilde{\mathbf{e}}_j$ can be arbitrarily far from $\tilde{\mathbf{b}}_j$ and $\tilde{\mathbf{c}}_j$ because the averaging process embodied in the kernel $\tilde{\mathbf{x}}$ is non-local and in that case $S^{intuitive} \neq S$.

References

Gradshteyn, IS and IM Ryzhik, Table of Integrals, Series and Prodicts, Corrected and Enlarged Edition, Academic Press, 1980.

Welch, PD. The Use of Fast Fourier Transform for the Estimation of Power Spectra: A Method Based on Time Averaging Over Short, Modified Periodograms." IEEE® Transactions on Audio and Electroacoustics. Vol. AU-15, 1967, pp. 70–73.

MatLab Script example2.m

```
% Relationship between
% the Fourier Transform (FT) of a long timeseries of length N
% (with frequency spacing Dw)
% and the FT of two shorter timeseries of length N/2
% (with frequency spacing 2*Dw)
% formed by cutting the longer timeseries in half
% Bill Menke, July 21, 2015
clear all;

N=256; % long dataset
M=N/2; % short datasets = half of long dataset
N4=N/4;

% random long dataset and its Fourier Transform
a = random('Normal',0,1,N,1);
atilde = fft(a);

% short datasets are left and right halves of long one
b = a(1:M);
c = a(M+1:N);
% their Fourier transforms
btilde = fft(b);
ctilde = fft(c);

% filter that implements a half-sample shift in frequency
xtilde = ones(M,1) - i*cot(pi*([0:M-1]'+0.5)/M);

Sxtilde = abs(xtilde);
figure(1);
clf;
set(gca,'LineWidth', 2);
hold on;
axis( [0-N/4, M-1-N/4, -max(Sxtilde), max(Sxtilde) ] );
plot( [0:M-1]-N/4, circshift(real(xtilde),N/4), 'r.','LineWidth', 2)
plot( [0:M-1]-N/4, circshift(imag(xtilde),N/4), 'b.','LineWidth', 2)
```

```
% convolution in frequency
% equivalent to phase shift in time
% equivalent to a Dw/2 shift in frequency
dtilde=cconv( xtilde, btilde, M)/M;
etilde=cconv( xtilde, ctilde, M)/M;

% Now construct the Fourier transform of the
% left and right long timeseries from the Fourier
% transform of the short pieces.  Note that the
% (even, odd) frequencies of the long time series
% are built differently.
ftilde = zeros(N,1);
gtilde = zeros(N,1);
for k=[1:M]
    ftilde(2*k-1) = btilde(k);
    ftilde(2*k) = dtilde(k);
    gtilde(2*k-1) = ctilde(k);
    gtilde(2*k) = -etilde(k);
end
arrtilde = ftilde+gtilde;

v = max( [abs(real(atilde)); abs(imag(atilde))] );
figure(2);
clf;
subplot(2,1,1);
set(gca,'LineWidth', 2);
hold on;
axis( [0, N-1, -v, v ] );
plot( [0:N-1], real(atilde), 'r-', 'LineWidth', 3);
plot( [0:N-1], real(arrtilde), 'g-', 'LineWidth', 2);
subplot(2,1,2);
set(gca,'LineWidth', 2);
hold on;
axis( [0, N-1, -v, v ] );
plot( [0:N-1], imag(atilde), 'r-', 'LineWidth', 3);
plot( [0:N-1], imag(arrtilde), 'g-', 'LineWidth', 2);

arr = real(ifft(arrtilde));
figure(3);
clf;
set(gca,'LineWidth',2);
hold on;
axis( [0, N-1, -max(abs(a)), max(abs(a)) ] );
plot( [0:N-1]', a, 'r-', 'LineWidth', 3);
plot( [0:N-1]', arr, 'g-', 'LineWidth', 2);
```